# SIMULATION OF UNDERWATER SOUND PROPAGATION IN A GENERAL STRATIFIED ENVIRONMENT BY A COUPLED MODE AND A FINITE ELEMENT METHOD

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**ABSTRACT**
A coupled mode and a finite element method are used in numerical simulations of underwater sound propagation in axially symmetric, multilayered environments with penetrable bottom and a number of fluid layers of different acoustic properties, separated by interfaces of general topography. The schemes are compared and applied to test problems with sea-mount type interfaces, and to sloping interface environments.

## 1. INTRODUCTION

We consider the underwater acoustic boundary-value problem (b.v.p.) in an axially symmetric sea environment characterized by a penetrable bottom of variable topography and several interfaces, also of general shape, separating fluid layers of different acoustic properties. The approximate solution of this waveguide b.v.p. is a central problem in computational underwater acoustics, [JKPS]. In this note, we shall use a coupled-mode and a finite element method for solving it.

The coupled-mode method, [ABL], [ABG], [AB], is effected by an enhanced local mode representation, which contains, in addition to the usual propagating and evanescent modes, additional correction modes, associated with the sloping interfaces and bottom. By including these sloping interface modes, one avoids using 'staircase' approximations resulting in loss of accuracy and energy or leading to heavy computational requirements. The finite element method, cf. [KD], [MKD], is a standard Galerkin/$P_1$ discretization of the b.v.p., coupled with an exact, nonlocal absorbing boundary condition at the exterior boundary of the waveguide and with an efficient iterative method for solving the attendant large sparse linear systems.

In Section 2 we pose the b.v.p. and introduce notation. In Sections 3 and 4 we present the two solvers in more detail, while in Section 5 we show numerical results that we obtained by applying the methods to two test cases of propagation and scattering in two-layer domains separated by sea-mount and upslope types of interfaces.

## 2. THE BOUNDARY-VALUE PROBLEM

We consider the range-dependent, cylindrically symmetric marine environment shown in Fig.1. For simplicity, we consider two fluid layers, water of constant density $r_1$ and sediment of constant density $r_2 > r_1$, separated by the interface $J : z = -h(r)$ and overlying a perfectly rigid horizontal boundary at $z = -H$. We let $c = c(r, z)$ be the speed of sound (discontinuous at the interface) and suppose that in the near region $D^N (0 \le r \le r_N)$ and the far region $D^F (r \ge r_F)$ the acoustic and geometric parameters are range independent. (Thus, $c$ and $h$ vary with $r$ only in

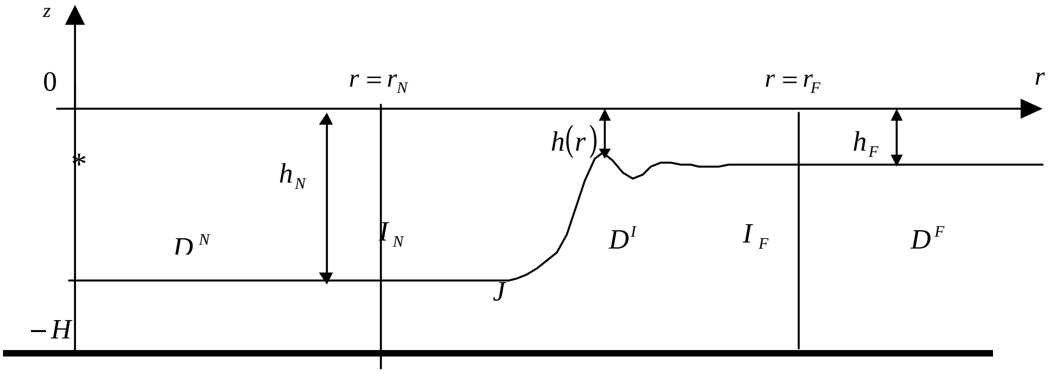


Figure 1. Domain decomposition and notation. The source is denoted by (*).

the intermediate region $D^I$, where $r_N < r < r_F$ .) The acoustic field is generated by a point harmonic source of frequency $f$, located at $z = z_0$ on the axis of symmetry. The acoustic propagation and scattering boundary-value problem in the domain $0 < r$, $-H \le z \le 0$, is to determine a complex-valued function $p = p(r,z)$ satisfying

$$\Delta p + k^2(r,z)\; p = -\frac{1}{2\pi}\frac{\delta(r)}{r}\delta(z - z_0), \qquad p(r,0) = 0, \qquad \frac{\partial p}{\partial z}(r,-H) = 0\,, \qquad \text{(1), (2), (3)}$$

the interface conditions

$$p\left(r,-h(r)^+\right) = p\left(r,-h(r)^-\right), \qquad \frac{1}{\rho_1}\frac{\partial p}{\partial n}\left(r,-h(r)^+\right) = \frac{1}{\rho_2}\frac{\partial p}{\partial n}\left(r,-h(r)^-\right), \qquad \text{(4), (5)}$$

and the radiation condition,

$$p(r,z) \sim outgoing cylindrical waves as r \to \infty\ . \qquad (6)$$

In (1) we have introduced the wave number $k = k(r,z) = \frac{2\pi f}{c}$; in (5) $\frac{\partial}{\partial n}$ denotes the normal derivative to the interface $z = -h(r)$ .

## 3. THE FINITE ELEMENT METHOD (FEM)

We consider the Helmholtz equation (1) with zero right hand side on the domain $\Omega = \{(z,r), -H \le z \le 0, R_1 \le r \le R_2\}$, where $0 < R_1 \le r_N$, $R_2 \ge r_F$ . At the left boundary $r = R_1$ we suppose that the field is given (by the coupled mode program), and at $z = 0$ and $z = -H$ we pose the b.c. (2) and (3), respectively. At the outer boundary $r = R_2$ we pose the exact, nonlocal, nonreflecting boundary condition (cf. [G] and the references in [KD])

$$\frac{\partial p}{\partial r} = T(p), \qquad r = R_2, -H \le z \le 0, \qquad . \qquad (7)$$

where $T$ is the integral operator associated with the DtN map of the exterior wave field evaluated at $r = R_2$, and given by $(Tp)(z) = \sum_{n=1}^{\infty} c_n(p) Z_n^F(z),$ where $c_n(p)$ = $k_n^F(s'_n/s_n)(p(\cdot,R_2),Z_n^F)_r$, $s'_n = \frac{dH_0^{(1)}}{dr}(k_n^F R_2)$, $s_n = H_0^{(1)}(k_n^F R_2)$, $(w,u)_r = \int_{-h_F}^{0} w\bar{u}dz + r\int_{-H}^{-h_F} w\bar{u}dz$, $r = \rho_1/\rho_2$, and $(k_n^F)^2$, $Z_n^F$ are the eigenvalues and eigenfunctions of the two-point depth eigenvalue problem $w'' + ((k^F)^2 - (k_n^F)^2)w = 0$ on $[-H,0]$, where $w(0) = w'(-H) = 0$, $w(-h_F^+) = w(-h_F^-)$, $w'(-h_F^+) = r w'(-h_F^-)$, and $k^F$ is the $z$-dependent wavenumber at any $r \ge r_F$ . We discretize this b.v.p. on $\Omega$ by the standard Galerkin/finite element method with continuous, piecewise linear functions defined on a triangulation of $\Omega$ with nodes on the interface $J$ . The nonlocal b.c. (7) becomes a generalized natural b.c. at $r = R_2$ and is approximated by a discrete analog evaluated as a finite sum of all the propagating and the most significant evanescent modes. The method has been shown to be second order accurate in the spatial discretization parameter, [G], [M]. The basic finite element module is incorporated in the Fortran code FENL described in detail in [KD] (http://oalib.saic.com/Other/fenl). The code uses mesh generation techniques from the Modulef library, and appropriate preconditioned iterative solvers for indefinite, sparse, complex linear systems from QMRPACK, [FN].

## 4. THE CONSISTENT COUPLED MODE METHOD (CCMM)

The problem (1)-(6) can be reformulated as a transmission problem in the bounded subdomain $D^I$ with the aid of the following general representations of the acoustic field in $D^N$ and $D^F$, respectively,

$$p^N = \frac{i}{4\rho_1}\sum_{n=1}^{\infty} Z_n^N(z_0)\, Z_n^N(z) H_0^{(1)}\left(k_n^N r\right) + \sum_{n=1}^{\infty} C_n^N Z_n^N(z) J_0\left(k_n^N r\right), \quad p^F = \sum_{n=1}^{\infty} C_n^F Z_n^F(z) H_0^{(1)}\left(k_n^F r\right), \tag{8}$$

and by requiring the matching of the field and its normal derivative at the common vertical interfaces $I_N$ and $I_F$. In formulas (8), the sets of numbers $\left\{k_n^N\right\}_{n=1,2,..}$ and $\left\{k_n^F\right\}_{n=1,2,..}$, and the sets of functions of $\left\{Z_n^N(z)\right\}_{n=1,2,..}$ and $\left\{Z_n^F(z)\right\}_{n=1,2,..}$, are the eigenvalues and eigenfunctions, respectively, of Sturm-Liouville problems, obtained by separation of variables in the subdomains $D^N$ and $D^F$. More details about the associated depth problem, and its solution in the case of two homogeneous layers: $\rho(0<z<-h)=\rho_1$, $c(r,0<z<-h)=c_1$, $\rho(-H<z<-h)=\rho_2$, $c(r,-H<z<-h)=$ $=c_2$, can be found in [B]. The transmission problem admits a variational formulation, expressed by the stationarity of the functional, [AB],

$$\boldsymbol{F}\left(p^I, \left\{C_n^N\right\}_{n\in N}, \left\{C_n^F\right\}_{n\in N}\right) = \sum_{\ell=1}^{2}\left\{ \frac{1}{2\rho_\ell}\int_{D_\ell^I}\left\{\left(\nabla p^I\right)^2 - k^2\left(p^I\right)^2\right\} dV + \frac{1}{\rho_\ell}\int_{I_\ell^N}\left(p^I - \frac{1}{2}p^N\left(\left\{C_n^N\right\}_{n\in N}\right)\right)\frac{\partial p^N\left(\left\{C_n^N\right\}_{n\in N}\right)}{\partial r} dS + \right.$$
$$\left. -\frac{1}{\rho_\ell}\int_{I_\ell^F}\left(p^I - \frac{1}{2}p^F\left(\left\{C_n^F\right\}_{n\in N}\right)\right)\frac{\partial p^F\left(\left\{C_n^F\right\}_{n\in N}\right)}{\partial r} dS \right\} - \frac{1}{2\rho_1}\sum_{n=1}^{\infty} C_n^N Z_n^N(z_0). \tag{9}$$

The variational principle, $\delta\boldsymbol{F}=0$, can be used to obtain an alternative, semi-discrete (Kantorovich) formulation of the problem in terms of local modes. This family of local basis functions is obtained by formulating and solving local, vertical Sturm-Liouville problems in the interval $[-H,\ 0]$. The enhanced local-mode representation of the acoustic field $p^I(r,z)$ in the variable-bathymetry/interface domain $D^I$, developed in [AB], reads as follows

$$p^I(r,z) = P_0(r) Z_0(z;r) + \sum_{n=1}^{\infty} P_n(r) Z_n(z;r). \tag{10}$$

where $P_n(r)$ denote the amplitudes of the modes, and the functions $Z_n(z;r)$, $n\geq 1$, are obtained as the eigenfunctions of the following *local, vertical eigenvalue* problem (defined for each $r^N < r < r^F$):

$$\frac{\partial^2 Z_n(z;r)}{\partial z^2} + \left(k^2(r,z) - k_n^2(r)\right) Z_n(z;r) = 0, \ -H\leq z\leq 0, \quad Z_n(0;r)=0, \quad \frac{\partial Z_n}{\partial z}(-H;r)=0, \tag{11}$$

in conjunction with the matching-interface conditions

$$Z_n\left(-h(r)^+;r\right) = Z_n\left(-h(r)^-;r\right), \qquad \frac{1}{\rho_1}\frac{\partial Z_n}{\partial z}\left(-h(r)^+;r\right) = \frac{1}{\rho_2}\frac{\partial Z_n}{\partial z}\left(-h(r)^-;r\right). \tag{12}$$

However, the local eigenfunctions $Z_n(z;r)$, $n\geq 1$, are incompatible with the sloping interface condition (5), whenever $dh(r)/dr \neq 0$. To remedy this inconsistency an additional mode is introduced in [AB], denoted by $P_0(r) Z_0(z;r)$ and called the *sloping-interface mode*. The vertical structure of the sloping-interface mode, $Z_0(z;r)$, is a continuous function satisfying the following conditions

$$Z_0(r)=0\,,\quad \frac{dZ_0(-H)}{dz}=0\,,\quad Z_0\left(-h(r)^{+};r\right)=Z_0\left(-h(r)^{-};r\right),\quad \frac{1}{\rho_1}\frac{\partial Z_n}{\partial z}\left(-h(r)^{+};r\right)-\frac{1}{\rho_2}\frac{\partial Z_n}{\partial z}\left(-h(r)^{-};r\right)=1. \tag{13}$$

In the series expansion (10), the first $0<n\le N$ terms $\{P_n(r)\;Z_n(z;r)\}_{n=1,2,..N}$, corresponding to real horizontal eigenvalues ($k_n^2>0$), are the propagating modes, and the terms $\{P_n(r)Z_n(z;r)\}$, $n=N+1,N+2,..$, corresponding to imaginary eigenvalues ($k_n^2<0$), are the evanescent modes. The sloping-interface mode $P_0(r)Z_0(z;r)$ is not needed when the interface is flat. Each term in the expansion (10) satisfies the free surface condition (2), the boundary condition (3) and the interface condition (4), individually. Thus, representation (10) renders all of them essential conditions in relation with the variational formulation. Using (10) in the variational principle, we obtain the following coupled-mode system of second-order ordinary differential equations, with respect to the mode amplitudes:

$$\sum_{n=0}^{\infty} a_{mn}(r)\frac{d^2P_n(r)}{dr^2}+b_{mn}(r)\frac{dP_n(r)}{dr}+c_{mn}(r)\,P_n(r)=0,\qquad m=0,1,2,3...\;, \tag{14}$$

where all coefficients are defined in terms of $Z_n(z;r)$ in $r_N<r<r_F$. The system (14) contains an additional equation, associated with the additional sloping-interface mode, and produces solutions consistent with the interface conditions and the conservation of energy. Eq. (14) is supplemented by the following end conditions

$$P_0(r_N)=P_0(r_F)=0,\; P_0'(r_N)=P_0'(r_F)=0,\; P_n'(r_N)+A_nP_n(r_N)=B_n\,,\quad P_n'(r_F)+D_nP_n'(r_F)=0\;, \tag{15}$$

$n=1,2,3...$, where the coefficients $A_n, B_n, D_n$ are given by

$$A_n=k_n^N\frac{J_1(k_n^Nr_N)}{J_0(k_n^Nr_N)},\; B_n=-\frac{iZ_n^N(z_0)k_n^N}{4\rho_1}\left(H_1^{(1)}(k_n^Nr_N)-H_1^{(1)}(k_n^Nr_N)\frac{J_1(k_n^Nr_N)}{J_0(k_n^Nr_N)}\right),\; D_n=k_n^F\frac{H_1^{(1)}(k_n^Fr_F)}{H_0^{(1)}(k_n^Fr_F)},\, n=1,2,3.... \tag{16}$$

## 5. NUMERICAL TESTS AND CONCLUSIONS

Extensive comparisons between FEM and CCMM, in the case of environments with a flat interface, have shown that both methods provide results in perfect agreement with the analytical solution, [B]. In this work we shall focus on two test cases, corresponding to a seamount and an upslope environment with very steep bathymetry, defined, respectively, by

$$h(r)=\begin{cases}50-25\cos\left(2\pi\dfrac{r-500}{400}\right) & \text{in } 300<r<700\\ 75, & \text{elsewhere}\end{cases}\qquad\text{and}\qquad h(r)=\begin{cases}50-25\tanh\left(3\pi\dfrac{r-500}{400}\right) & \text{in } 300<r<700\\ 75,\; r\le 300; & 25,\; r\ge 700\end{cases},$$

where all distances are in meters. In both cases the density and sound speed of the seawater are taken constant $\rho_1=1gr/cm^3$, $c_1=1500$ m/s, and the density and sound speed of the sediment $\rho_2=1.5gr/cm^3$ and $c_2=1700$ m/s. The source frequency is taken to be 25Hz. In the first case, presented in Figs. 2 and 3, the pulsating source is located at $z_0=-70\mathrm{m}$ (near the interface), and in the second case, presented in Figs. 4 and 5, at $z_0=-25\mathrm{m}$ (near the free surface). The number of propagating modes in $D^N$ is $N$=3. Numerical results concerning the Transmission Loss (TL in dB), as calculated by CCMM and FEM, respectively, are compared in Figs. 2 and 4. In the case of the sea-mount, comparisons of the Transmission Loss, at receiver's depths RD=50m and at RD=SD=70m, are presented in Fig.3. In the case of the upslope environment, comparisons of the Transmission Loss at RD=SD=25m and at RD=90m are presented in Fig.5. We can observe from these figures that the agreement between the two methods is very good, in the whole domain, although the computational requirements of the FEM, as compared to CCMM, are significantly larger. On the other hand, the FEM, is inherently more flexible to treat localized inhomogeneities. Thus, after further comparison and validation, FEM and CCMM can be used to complement each other, in order to treat difficult situations, such as acoustic scattering problems from localized scatterers embedded in nonhomogeneous waveguides.

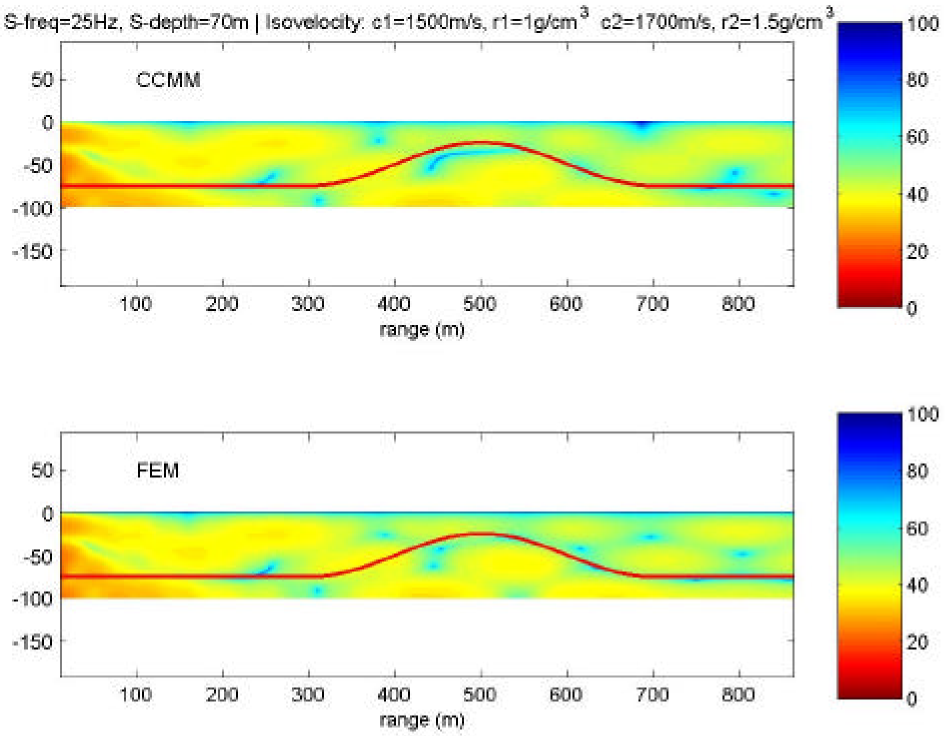


Fig.2 Comparison of FEM and CCEM in the case of a seamount. Transmission Loss (in dB).

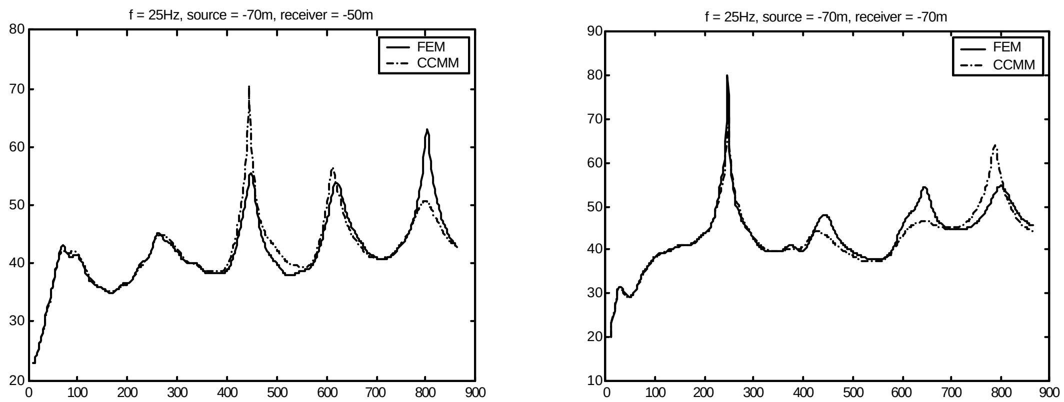


Fig.3 Transmission Loss (in dB) at RD=50m and at RD=SD=70m in the case of a seamount

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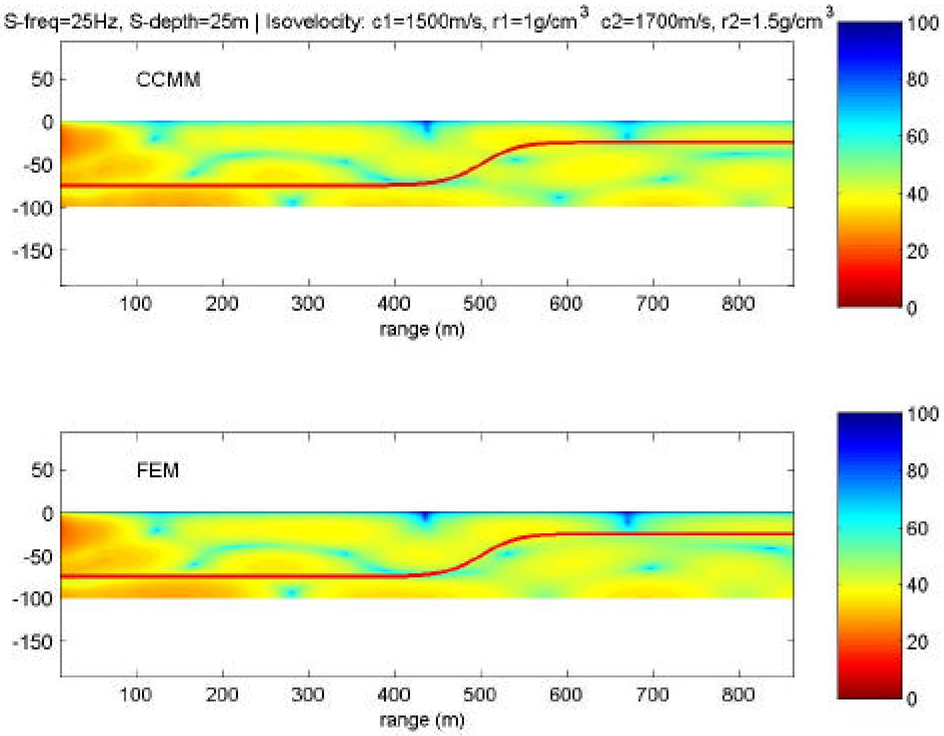


Fig.4 Comparison of FEM and CCEM in the case of an upslope environment. Transmission Loss (in dB).

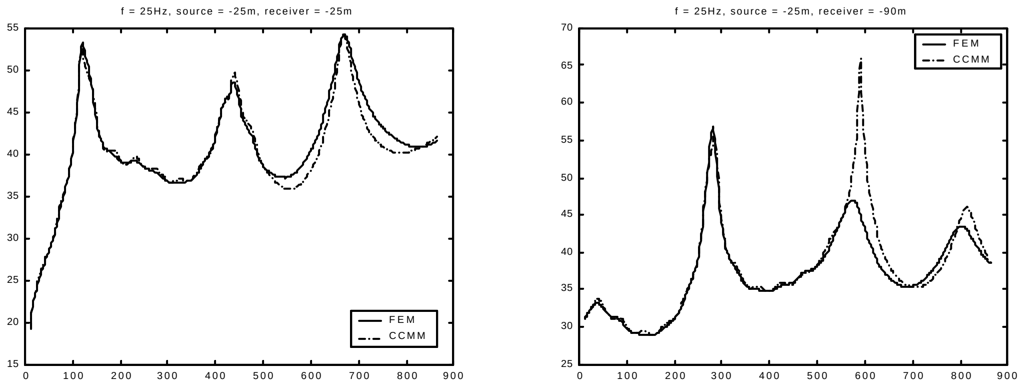


Fig.5 Transmission Loss (in dB) at SD=RD=25m and at RD=90m in the case of a seamount


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