HELMHOLTZ EQUATION WITH ARTIFICIAL BOUNDARY CONDITIONS IN A TWO-DIMENSIONAL WAVEGUIDE [*]

D.A. MITSOUDIS [† §], CH. MAKRIDAKIS [‡ §], AND M. PLEXOUSAKIS [‡ §]

Abstract. We consider a time-harmonic acoustic wave propagation problem in a two dimensional water waveguide confined between a horizontal surface and a locally varying bottom. We formulate a model based on the Helmholtz equation coupled with nonlocal Dirichlet-to-Neumann boundary conditions imposed on two artificial boundaries. We establish the well-posedness of the associated variational problem, under the assumption of a downsloping bottom, by showing stability estimates in appropriate function spaces. The outcome of some numerical experiments with a code implementing a standard/Galerkin finite element approximation of the variational formulation of the model are also presented.

Key words. Helmholtz equation, waveguide, nonlocal boundary conditions, *a priori* estimates.

AMS subject classifications. 35J05, 35J20, 65N30, 76Q05

1. Introduction.

In this paper we develop and analyze a model for wave propagation based on the Helmholtz equation in the context of a realistic environment widely used in applications, especially in underwater acoustics. In direct acoustic propagation and scattering applications the Helmholtz equation is a model, in the frequency domain, of the sound propagation and backscattering caused, usually, by a point source which emits a continuous time-harmonic signal. Such models have been extensively analyzed in the past, but in most of the cases the formulation of boundary value problems was based on certain simplifying assumptions mainly for the domain and the boundary conditions.

Here, we consider a two-dimensional waveguide in Cartesian coordinates consisting of a homogeneous water column confined between a horizontal pressure-release sea surface and an acoustically soft sea floor. The original infinite domain is truncated with two artificial boundaries and we formulate a model in the resulting bounded domain by introducing suitable nonlocal conditions on the artificial boundaries. The proposed model simulates efficiently the effect of the source and the backscattered field from the rest of the waveguide and is appropriate for finite element computations. The main task in this paper is to show the well-posedness of the model. The challenging technical difficulties which usually arise in the analysis of Helmholtz-type equations are, of course, present in our case too. In addition, the nonlocal nature of the boundary conditions considered herein introduces nontrivial complications in the analysis. We show stability estimates in appropriate Sobolev norms which, in turn, imply existence and uniqueness of the solution. The estimates involve constants with explicit dependence on the wavenumber and the geometrical parameters of the problem.

Problem description and results. We assume that the waveguide may be broken into three parts: a) a semi-infinite strip of constant depth, where the source is located, b) a bounded intermediate region, where the bottom may vary, and c) another semi-infinite strip, also of constant depth; the exact setting is described in Section 2. Despite the fact that certain simplifying

---

[†]Archimedes Center for Modeling, Analysis & Computation (ACMAC), Department of Applied Mathematics, University of Crete, Heraklion 71003, Greece.

[‡]Department of Applied Mathematics, University of Crete, P.O. Box 2208, Heraklion 71409, Greece.

[§]Institute of Applied and Computational Mathematics, FORTH, Heraklion 71110, Greece.

[*]This work was partly supported by the FP7-REGPOT-2009-1 project "Archimedes Center for Modeling, Analysis and Computation" funded by the European Commission.

assumptions are made, this model still exhibits many of the basic features associated with ocean acoustic propagation, [19]. On the other hand, this type of scattering problem is mathematically challenging, mainly because of the unboundedness of the environment. We also emphasize the fact that such models may serve as the basis of developing direct, efficient numerical methods, and thus are important in computational wave propagation; see, *e.g.*, [19, Chapter 5]. In fact, in the course of developing direct numerical methods for such problems there emerges the need of the appropriate truncation of the infinite domain and the reformulation of the problem as a boundary value problem posed in a bounded computational domain. Many methods have been proposed in order to reduce the originally infinite domain into a bounded one. These include the introduction of artificial boundaries on which local or nonlocal absorbing boundary conditions are imposed (see, *e.g.*, the review paper of Tsynkov [32], the book by Givoli [14], [18], [17], and the references therein) and the use of perfectly matched layers (see, for example, [31, 3]).

In the present paper, the infinite domain described earlier is truncated by introducing two artificial boundaries, one 'near' the source and one far from the source, which bound the part of the domain that supports the variable part of the bottom. On these boundaries we impose nonlocal Dirichlet–to–Neumann (DtN) type boundary conditions: far from the source a classical DtN outgoing boundary condition which, to the best of our knowledge, was introduced in the context of underwater acoustics applications in [13] (see also [15], [4]), and near the source a nonhomogeneous DtN-type boundary condition, which was proposed and coupled with a finite element method in [26], for a cylindrically symmetric waveguide consisting of multiple fluid layers with different acoustic parameters. This method was implemented in a finite element code which has been extensively tested and proved to compare very well with other established codes in the underwater acoustics community; see [26], [1]. Nevertheless, the corresponding model has not been theoretically analyzed until now. This is the task of the present work.

The rest of the paper is organized as follows. In Section 3 we describe the model and its weak formulation and we introduce an appropriate functional space setting.

Section 4 concerns the well-posedness of the variational problem and is divided in three parts. In §4.1 well-posedness is established under the assumption of a downsloping bottom, and an additional assumption on the location of the artificial boundary near the source. *A priori* estimates involving explicit dependence on the frequency and the geometrical parameters of the problem are derived. The analysis is based on the possibility of using appropriate test functions involving the first order weak derivatives of the solution in the bilinear form and the careful treatment of the nonlocal boundary terms. Such test functions have the property of enhancing the bilinear form of the Helmholtz operator to a positive principal part and, thus, make the derivation of *a priori* estimates in L^2 and H^1 possible. On the other hand their presence in the boundary terms introduces nontrivial complications in the analysis which are successfully addressed herein. Such functions have been used in the past in the analysis of the well-posedness of Helmholtz equations in [23], [24], [10], [9], [25], and were first introduced to derive an important identity for the Helmholtz operator in [28].

In §4.2 we show how the assumption regarding the position of the near-field boundary may be relaxed. This section closes with §4.3, where we prove existence-uniqueness under a stringent 'small-frequency/shallow-water' assumption.

In Section 5 we present, as a proof of concept, the outcome of some numerical tests that we have performed with a finite element code, which discretizes our model with a standard/Galerkin finite element scheme based on piecewise linear basis functions. Interesting conclusions on the sensitivity of the model with respect to the geometrical setup and its efficiency are derived.

2. Formulation of the boundary value problem. In this section we describe the geometric configuration, introduce basic notation, and define the boundary value problem that we deal with in what follows. We consider a two-dimensional Cartesian waveguide consisting of a single water layer confined between a horizontal pressure-release surface and a (locally) varying, acoustically soft bottom; see Figure 2.1. We assume that the sound speed c_0, as well as the density, are constant in the water layer. A Cartesian coordinate system (x, y) is introduced with its x-axis lying on the free surface and the depth coordinate y being positive downward. The acoustic field is generated by a time-harmonic point source of frequency f located at $(x_{\mathrm{s}}, y_{\mathrm{s}})$. (Typically, in ocean acoustics applications this source is referred to as a line source [19].) The wavelength is being denoted by $\lambda := c_0/f$ and the constant (real) wavenumber is $k = 2\pi/\lambda$. The bottom is prescribed by a sufficiently smooth, positive function h of the form

$$h(x) = \begin{cases} D_{\mathrm{N}} & \text{for } x \leq x_1, \\ h_{\mathrm{b}}(x) & \text{for } x_1 < x < x_2, \\ D_{\mathrm{F}} & \text{for } x \geq x_2, \end{cases}$$

where $x_1 > x_{\mathrm{s}}$ and D_{N}, D_{F} are positive constants.

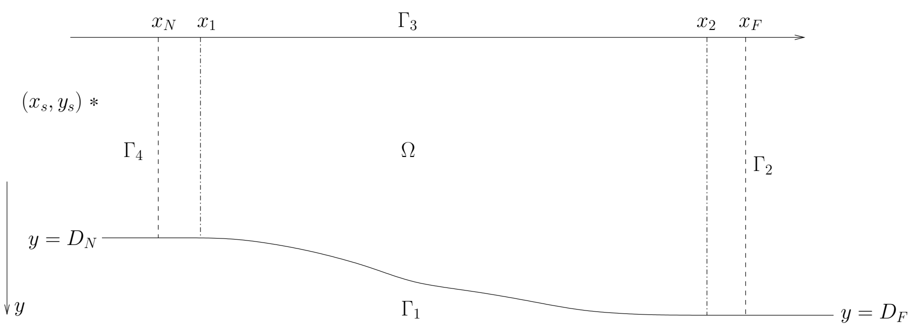

FIG. 2.1. *Schematic representation of the waveguide and basic notation. The position of the point source is marked with an asterisk.*

In what follows, we concentrate on studying the acoustic propagation and scattering problem in the semi-infinite part of the waveguide which supports the bottom topography irregularities, *i.e.*, for $x \geq x_{\mathrm{s}}$. In this environment the acoustic field (usually acoustic pressure) satisfies the Helmholtz equation [19],

$$-\Delta u(x, y) - k^2 u(x, y) = \delta(x - x_{\mathrm{s}})\delta(y - y_{\mathrm{s}}), \tag{2.1}$$

where δ denotes the Dirac distribution. Equation (2.1) is supplemented by homogeneous Dirichlet boundary conditions on the surface and on the bottom and by an appropriate radiation condition stating that

$$u(x, y) \text{ is 'outgoing' as } x \to +\infty.$$

When one is interested in solving this problem computationally with a direct numerical method, the original infinite domain has to be truncated. One way to achieve this is by introducing two artificial boundaries at some appropriate values of x, near the source (at $x = x_{\mathrm{N}} \in (x_{\mathrm{s}}, x_1)$) and far from the source (at $x = x_{\mathrm{F}} > x_2$). On these artificial boundaries suitable nonlocal conditions of DtN type may then be imposed, which are essentially derived from explicit solutions of the associated PDE problem in the near-field ($x_{\rm s} < x < x_1$) and far-field ($x > x_2$) regions. Next, we derive these conditions by distinguishing cases with respect to far-field and near-field regions.

2.1. Far-field region. The outgoing acoustic field in the far-field region, *i.e.*, for $x > x_2$, may be written as (compare to [15])

$$u^{\rm F}(x,y) = \sum_{n=1}^{M_{\rm F}} c_n\, {\rm e}^{i\sqrt{k^2-\mu_n^{\rm F}}x}\, Y_n^{\rm F}(y) + \sum_{n=M_{\rm F}+1}^{\infty} c_n\, {\rm e}^{-\sqrt{\mu_n^{\rm F}-k^2}x}\, Y_n^{\rm F}(y), \tag{2.2}$$

where $\{\mu_n^{\rm F}\}_{n\geq 1}$ is the increasing sequence of eigenvalues and $\{Y_n^{\rm F}\}_{n\geq 1}$ the corresponding orthonormal eigenfunctions of the vertical eigenvalue problem

$$\frac{d^2 Y_n^{\rm F}}{dy^2} + \mu_n^{\rm F} Y_n^{\rm F} = 0 \quad \text{in } [0, D_{\rm F}], \tag{2.3}$$

$$Y_n^{\rm F}(0) = Y_n^{\rm F}(D_{\rm F}) = 0. \tag{2.4}$$

$M_{\rm F}$ is an index for which

$$\mu_{M_{\rm F}} < k^2 < \mu_{M_{\rm F}+1}. \tag{2.5}$$

Stated otherwise, the first $M_{\rm F}$ terms in (2.2) correspond to the so-called *propagating* modes, while the rest of them correspond to the *evanescent* modes. Note that in our case, where k is constant, the eigenvalues and the corresponding orthonormal eigenfunctions are simply

$$\mu_n^{\rm F} = \left(\frac{n\pi}{D_{\rm F}}\right)^2, \quad Y_n^{\rm F} = \sqrt{\frac{2}{D_{\rm F}}}\,\sin\sqrt{\mu_n^{\rm F}}y = \sqrt{\frac{2}{D_{\rm F}}}\,\sin\frac{n\pi y}{D_{\rm F}},\ n = 1, 2, \ldots,$$

respectively.

The $Y_m^{\rm F}$, $m = 1, 2, \ldots$, form a complete orthonormal system in $L^2(0, D_{\rm F})$ with respect to the standard inner product; therefore the coefficients c_n in (2.2) satisfy, for each $\tilde{x} > x_2$,

$$c_m = \begin{cases} u_m^{\rm F}(\tilde{x})\,{\rm e}^{-i\sqrt{k^2-\mu_m^{\rm F}}\,\tilde{x}}, & m \leq M_{\rm F}, \\ u_m^{\rm F}(\tilde{x})\,{\rm e}^{\sqrt{\mu_m^{\rm F}-k^2}\,\tilde{x}}, & m \geq M_{\rm F}+1, \end{cases} \tag{2.6}$$

where

$$u_m^{\rm F}(\tilde{x}) := \int_0^{D_{\rm F}} u^{\rm F}(\tilde{x}, y)\, Y_m^{\rm F}(y)\, dy \tag{2.7}$$

are the Fourier coefficients of the restriction of $u^{\rm F}$ on $\{(x,y) : x = \tilde{x}\}$. Now, considering a $x_{\rm F} > x_2$ let us denote by $\Gamma_2 := \{(x,y) : x = x_{\rm F},\ y \in [0, D_{\rm F}]\}$ the (artificial) far-field boundary. Then, the DtN map of the acoustic field for $x > x_2$ evaluated on Γ_2 is simply the matching condition

$$\frac{\partial u}{\partial x}(x_{\rm F}, y) = Tu(y) := T_1 u(y) + T_2 u(y), \tag{2.8}$$

where

$$T_1 u(y) := i \sum_{n=1}^{M_{\rm F}} \sqrt{k^2 - \mu_n^{\rm F}}\, u_n^{\rm F}(x_{\rm F})\, Y_n^{\rm F}(y) \tag{2.9}$$

and

$$T_2 u(y) := -\sum_{n=M_{\rm F}+1}^{\infty} \sqrt{\mu_n^{\rm F} - k^2}\, u_n^{\rm F}(x_{\rm F})\, Y_n^{\rm F}(y). \tag{2.10}$$

Notice that when discretizing the model the term involving the series in $T_2 u(y)$ should be truncated appropriately.

2.2. Near-field region. In order to derive a DtN-type nonlocal condition for an artificial boundary near the source we need an analytic expression of the acoustic field for the region near the source. In the case of a cylindrically symmetric environment, a normal-mode representation of the field can be found in [12]. For $x \in (x_{\rm s}, x_1)$ the problem is also separable, so letting $\{\mu_n^{\rm N}, Y_n^{\rm N}\}_{n\geq 1}$ denote the (increasing) eigenvalues and the corresponding orthonormal eigenfunctions of the associated vertical eigenvalue problem

$$\frac{d^2 Y_n^{\rm N}}{dy^2} + \mu_n^{\rm N} Y_n^{\rm N} = 0 \quad \text{in } [0, D_{\rm N}], \tag{2.11}$$

$$Y_n^{\rm N}(0) = Y_n^{\rm N}(D_{\rm N}) = 0, \tag{2.12}$$

where $M_{\rm N}$ is a positive integer such that

$$\mu_{M_{\rm N}} < k^2 < \mu_{M_{\rm N}+1}, \tag{2.13}$$

we obtain the following series representation which involves both incoming and outgoing wave terms:

$$\begin{aligned} u^{\rm N}(x,y) &= \sum_{n=1}^{M_{\rm N}} \left(a_n\, {\rm e}^{i\sqrt{k^2-\mu_n^{\rm N}}(x-x_{\rm s})} + b_n\, {\rm e}^{-i\sqrt{k^2-\mu_n^{\rm N}}(x-x_{\rm s})} \right) Y_n^{\rm N}(y) \\ &\quad + \sum_{n=M_{\rm N}+1}^{\infty} \left(a_n\, {\rm e}^{-\sqrt{\mu_n^{\rm N}-k^2}(x-x_{\rm s})} + b_n\, {\rm e}^{\sqrt{\mu_n^{\rm N}-k^2}(x-x_{\rm s})} \right) Y_n^{\rm N}(y). \end{aligned} \tag{2.14}$$

On the other hand, the solution of (2.1) (propagating from left to right), if the waveguide were a strip of constant depth $D_{\rm N}$, would be given by the Green function (see, *e.g.*, [21])

$$\begin{aligned} u_{\rm out}(x,y) &= \frac{i}{2} \sum_{n=1}^{M_{\rm N}} \frac{1}{\sqrt{k^2-\mu_n^{\rm N}}}\, {\rm e}^{i\sqrt{k^2-\mu_n^{\rm N}}(x-x_s)}\, Y_n^{\rm N}(y_{\rm s})\, Y_n^{\rm N}(y) \\ &\quad + \frac{1}{2} \sum_{n=M_{\rm N}+1}^{\infty} \frac{1}{\sqrt{\mu_n^{\rm N}-k^2}}\, {\rm e}^{-\sqrt{\mu_n^{\rm N}-k^2}(x-x_s)}\, Y_n^{\rm N}(y_{\rm s})\, Y_n^{\rm N}(y). \end{aligned} \tag{2.15}$$

Next, assuming that as $x \downarrow x_{\rm s}$ the field (2.14) agrees asymptotically with the outgoing field (2.15) produced by the source in the part of the strip confined between $x_{\rm s}$ and x_1, we obtain the following relation between the coefficients a_n and b_n in (2.14):

$$a_n = \begin{cases} -b_n + \frac{i}{2\sqrt{k^2-\mu_n^{\rm N}}}\, Y_n^{\rm N}(y_{\rm s}), & n = 1, \dots, M_{\rm N}, \\ -b_n + \frac{1}{2\sqrt{\mu_n^{\rm N}-k^2}}\, Y_n^{\rm N}(y_{\rm s}), & n \geq M_{\rm N}+1. \end{cases}$$

Letting

$$\beta_n = \begin{cases} -2ib_n, & n = 1, \dots, M_{\rm N}, \\ 2b_n, & n \geq M_{\rm N}+1, \end{cases}$$

we finally conclude for $x \in (x_{\rm s}, x_1)$

$$
\begin{aligned}
u^{\rm N}(x,y) = & \sum_{n=1}^{M_{\rm N}} \beta_n \sin\Big(\sqrt{k^2-\mu_n^{\rm N}}(x-x_{\rm s})\Big)\, Y_n^{\rm N}(y) + \\
& + \sum_{n=M_{\rm N}+1}^{\infty} \beta_n \sinh\Big(\sqrt{\mu_n^{\rm N}-k^2}(x-x_{\rm s})\Big)\, Y_n^{\rm N}(y) \\
& + \frac{i}{2}\sum_{n=1}^{M_{\rm N}} \frac{1}{\sqrt{k^2-\mu_n^{\rm N}}}\, Y_n^{\rm N}(y_{\rm s})\, Y_n^{\rm N}(y)\, {\rm e}^{i\sqrt{k^2-\mu_n^{\rm N}}(x-x_s)} \\
& + \frac{1}{2}\sum_{n=M_{\rm N}+1}^{\infty} \frac{1}{\sqrt{\mu_n^{\rm N}-k^2}}\, Y_n^{\rm N}(y_{\rm s})\, Y_n^{\rm N}(y)\, {\rm e}^{-\sqrt{\mu_n^{\rm N}-k^2}(x-x_s)}.
\end{aligned}
\tag{2.16}
$$

Denote, as before,

$$
u_m^{\rm N}(\tilde{x}) := \int_0^{D_{\rm N}} u^{\rm N}(\tilde{x},y)\, Y_m^{\rm N}(y)\, dy. \tag{2.17}
$$

Then (2.16) and the orthonormality of $Y_m^{\rm N}$'s imply that

$$
\beta_m = \begin{cases} \dfrac{1}{\sin\Big(\sqrt{k^2-\mu_m^{\rm N}}(\tilde{x}-x_{\rm s})\Big)} \left(u_m^{\rm N}(\tilde{x}) - \dfrac{i}{2\sqrt{k^2-\mu_m^{\rm N}}}\, {\rm e}^{i\sqrt{k^2-\mu_m^{\rm N}}(\tilde{x}-x_{\rm s})}\, Y_m^{\rm N}(y_{\rm s}) \right), & m \le M_{\rm N}, \\ \dfrac{1}{\sinh\Big(\sqrt{\mu_m^{\rm N}-k^2}(\tilde{x}-x_{\rm s})\Big)} \left(u_m^{\rm N}(\tilde{x}) - \dfrac{1}{2\sqrt{\mu_m^{\rm N}-k^2}}\, {\rm e}^{-\sqrt{\mu_m^{\rm N}-k^2}(\tilde{x}-x_{\rm s})} Y_m^{\rm N}(y_{\rm s}) \right), & m \ge M_{\rm N}+1. \end{cases} \tag{2.18}
$$

We differentiate (2.16) with respect to x, evaluate the resulting expression at $x = x_{\rm N} \in (x_{\rm s}, x_1)$, and replace the coefficients β_m by (2.18) to get, after some calculations, that the nonlocal near-field condition on $\Gamma_4 := \{(x,y):\ x = x_{\rm N},\ y \in [0, D_{\rm N}]\}$ may be written in the form

$$
\frac{\partial u}{\partial x}(x_{\rm N}, y) = Ru(y) + S(y) := R_1 u(y) + R_2 u(y) + S_1(y) + S_2(y), \tag{2.19}
$$

where

$$
R_1 u(y) := \sum_{n=1}^{M_{\rm N}} \sqrt{k^2-\mu_n^{\rm N}}\, \cot\Big(\sqrt{k^2-\mu_n^{\rm N}}(x_{\rm N}-x_{\rm s})\Big)\, u_n^{\rm N}(x_{\rm N})\, Y_n^{\rm N}(y), \tag{2.20}
$$

$$
R_2 u(y) := \sum_{n=M_{\rm N}+1}^{\infty} \sqrt{\mu_n^{\rm N}-k^2}\, \coth\Big(\sqrt{\mu_n^{\rm N}-k^2}(x_{\rm N}-x_{\rm s})\Big)\, u_n^{\rm N}(x_{\rm N})\, Y_n^{\rm N}(y), \tag{2.21}
$$

$$
S_1(y) := -\frac{i}{2}\sum_{n=1}^{M_{\rm N}} \frac{1}{\sin\Big(\sqrt{k^2-\mu_n^{\rm N}}(x_{\rm N}-x_{\rm s})\Big)}\, Y_n^{\rm N}(y_{\rm s})\, Y_n^{\rm N}(y), \tag{2.22}
$$

$$
S_2(y) := -\frac{1}{2}\sum_{n=M_{\rm N}+1}^{\infty} \frac{1}{\sinh\Big(\sqrt{\mu_n^{\rm N}-k^2}(x_{\rm N}-x_{\rm s})\Big)}\, Y_n^{\rm N}(y_{\rm s})\, Y_n^{\rm N}(y). \tag{2.23}
$$

At this point let us note that $x_{\rm N}$ is assumed to be chosen appropriately to ensure that

$$
\sin\Big(\sqrt{k^2-\mu_n^{\rm N}}(x_{\rm N}-x_{\rm s})\Big) \neq 0 \quad \text{for } n = 1, \ldots, M_{\rm N}. \tag{2.24}
$$

In fact, as will be evident in the following sections, it is convenient for the subsequent analysis to choose x_{N} such that

$$\cot\left(\sqrt{k^2-\mu_n^{\mathrm{N}}}(x_{\mathrm{N}}-x_{\mathrm{s}})\right)>0 \quad \text{for } n=1,\dots,M_{\mathrm{N}}. \tag{2.25}$$

Since $\sqrt{k^2-\mu_1^{\mathrm{N}}}(x_{\mathrm{N}}-x_{\mathrm{s}}) > \cdots > \sqrt{k^2-\mu_{M_{\mathrm{N}}}^{\mathrm{N}}}(x_{\mathrm{N}}-x_{\mathrm{s}}) > 0$, it suffices to choose x_{N} such that $\sqrt{k^2-\mu_1^{\mathrm{N}}}(x_{\mathrm{N}}-x_{\mathrm{s}}) < \pi/2$. In our case, where k is constant, the eigenvalues are equal to $\mu_n^{\mathrm{N}}=(n\pi/D_{\mathrm{N}})^2$, $n=1,\dots,M_{\mathrm{N}}$, and it is easy to check that (2.25) is satisfied if we select x_{N} so that

$$x_{\mathrm{N}}-x_{\mathrm{s}}\le \lambda/4, \tag{2.26}$$

where λ is the wavelength. Notice that since our aim is the design of an appropriate artificial boundary condition, the choice of x_{N} is at our disposal and thus (2.26) is easily satisfied.

3. The model and its weak formulation. Let us denote by Ω the bounded part of the waveguide confined between $x=x_{\mathrm{N}}$ and $x=x_{\mathrm{F}}$ and, also, $\Gamma_1:=\{(x,y):\ x\in[x_{\mathrm{N}},x_{\mathrm{F}}],\ y=h(x)\}$ and $\Gamma_3:=\{(x,y):\ x\in[x_{\mathrm{N}},x_{\mathrm{F}}],\ y=0\}$. Thus, we formulate the following problem in the bounded domain Ω: We seek for a complex-valued function u such that

$$\begin{aligned} &-\Delta u-k^2u=0 \quad \text{in } \Omega,\\ &u=0 \quad \text{on } \Gamma_1\cup\Gamma_3,\\ &u_x=Tu \quad \text{on } \Gamma_2,\\ &u_x=Ru+S \quad \text{on } \Gamma_4, \end{aligned} \tag{3.1}$$

where the nonlocal operators T and R and the function S are defined in (2.8)–(2.10) and (2.19)–(2.23), respectively.

Before proceeding with the weak formulation of problem (3.1) we introduce some notation and the function space setting. We let $(\cdot,\cdot)_D$ denote the usual L^2-inner product in D, $\|\cdot\|_D$ the corresponding L^2-norm in D, while $\|\cdot\|_{m,D}$ denotes the standard Sobolev norm of $H^m(D)$. To deal with operators on the boundaries we shall use appropriate subspaces of $H^s(\Gamma_i)$ for $s=1/2,1$. To be more specific we introduce the space $X^s(\Gamma_i)$, $s\ge 0$, to be the subspace of $L^2(\Gamma_i)$ of functions admitting a representation in terms of the eigenfunctions Y_m^{E} of the Laplace–Beltrami operator on Γ_i with coefficients v_m^{E} satisfying

$$\|v\|_{X^s(\Gamma_i)}=\left(\sum_{m=1}^{\infty}(\mu_m^{\mathrm{E}})^s|v_m^{\mathrm{E}}(x_{\mathrm{E}})|^2\right)^{1/2}<\infty,$$

where $i=2$ or 4 and $\mathrm{E}=\mathrm{F}$ or N, respectively, depending on whether we lie on the far-field boundary Γ_2 or on the near-field boundary Γ_4. Here μ_m^{F} and μ_m^{N} are the eigenvalues of the vertical problems (2.3)– (2.4) and (2.11)– (2.12), respectively. The notation is adopted from [4].

Then (see [22], [16]), $X^s(\Gamma_i)$ coincides with $H^s(\Gamma_i)$ for $0<s<1/2$. For $s=1/2$, $X^{1/2}(\Gamma_i)$ may be identified with $H^{1/2}_{00}(\Gamma_i)$, the subspace of functions of $H^{1/2}(\Gamma_i)$ which when extended by zero belong to $H^{1/2}(\partial\Omega)$. For $1/2<s\le 1$, $X^s(\Gamma_i)=\overset{0}{H^s}(\Gamma_i)$.

The dual space of $X^s(\Gamma_i)$, denoted by $X^{-s}(\Gamma_i)$, may be identified with all sequences $\{v_m^{\mathrm{E}}\}_{n\ge1}$ such that $\sum_{n=1}^{\infty}(\mu_m^{\mathrm{E}})^{-s}|v_m^{\mathrm{E}}(x_{\mathrm{E}})|^2<\infty$. In that case $v=\sum_{n=1}^{\infty}v_m^{\mathrm{E}}Y_m^{\mathrm{E}}$ can be considered as an element of $X^{-s}(\Gamma_i)$ with norm

$$\|v\|^2_{X^{-s}(\Gamma_i)}=\sum_{n=1}^{\infty}(\mu_m^{\mathrm{E}})^{-s}|v_m^{\mathrm{E}}(x_{\mathrm{E}})|^2.$$

Now, let us define

$$\mathcal{H} = \{v : v \in H^1(\Omega) \text{ and } v = 0 \text{ on } \Gamma_1 \text{ and } \Gamma_3\}.$$

Then, the usual trace operators $u \mapsto u(x_{\rm N}, \cdot)$ and $u \mapsto u(x_{\rm F}, \cdot)$ are continuous mappings from $\mathcal{H}$ into $X^{1/2}(\Gamma_4)$ and $X^{1/2}(\Gamma_2)$, respectively.

We equip $\mathcal{H}$ with the wavenumber-dependent norm, [23], [9]

$$\|v\|_{\mathcal{H}} := \left(\|\nabla v\|^2_\Omega + k^2\|v\|^2_\Omega\right)^{1/2}.$$

Then a weak formulation of (3.1) is as follows: We seek $u \in \mathcal{H}$ such that

$$\mathcal{B}(u, v) = -(S, v)_{\Gamma_4} = -\int_{\Gamma_4} S\bar{v}\, dy \quad \text{for all } v \in \mathcal{H}, \tag{3.2}$$

where the sesquilinear form $\mathcal{B}$ is defined as

$$\mathcal{B}(u, v) := \int_\Omega \nabla u \cdot \nabla \bar{v} - k^2 \int_\Omega u\bar{v} - \int_{\Gamma_2} T(u)\bar{v}\, dy + \int_{\Gamma_4} R(u)\bar{v}\, dy. \tag{3.3}$$

The following lemma shows that $\mathcal{B}(\cdot, \cdot)$ is well defined on $\mathcal{H} \times \mathcal{H}$ and is, in fact, continuous.

LEMMA 3.1. *There exist constants C_α, C_β, and C_γ such that for all $u, v \in \mathcal{H}$*

$$|(Ru, v)_{\Gamma_4}| \le C_\alpha \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}, \tag{3.4}$$

$$|(Tu, v)_{\Gamma_2}| \le C_\beta \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}, \tag{3.5}$$

and

$$|(S, v)_{\Gamma_4}| \le C_\gamma \|S\|_{X^{-1/2}(\Gamma_4)} \|v\|_{\mathcal{H}}. \tag{3.6}$$

Proof. Let $u, v \in \mathcal{H}$. We begin by noting that R may be viewed as a linear map from $X^{1/2}(\Gamma_4)$ into $X^{-1/2}(\Gamma_4)$. Since $\{Y^{\rm N}_n\}_{n\ge1}$ forms an orthonormal basis in $L^2(0, D_{\rm N})$, Parseval's relation and (2.19)-(2.21) imply that

$$\begin{aligned} |(Ru, v)_{\Gamma_4}| \le {}& |\sum_{n=1}^{M_{\rm N}} \sqrt{k^2 - \mu^{\rm N}_n} \cot\left(\sqrt{k^2 - \mu^{\rm N}_n}(x_{\rm N} - x_{\rm s})\right) u^{\rm N}_n(x_{\rm N})\overline{v^{\rm N}_n(x_{\rm N})}| \\ &+ |\sum_{n=M_{\rm N}+1}^{\infty} \sqrt{\mu^{\rm N}_n - k^2} \coth\left(\sqrt{\mu^{\rm N}_n - k^2}(x_{\rm N} - x_{\rm s})\right) u^{\rm N}_n(x_{\rm N})\overline{v^{\rm N}_n(x_{\rm N})}|. \end{aligned}$$

Let

$$C_1 = \max_{1\le n\le M_{\rm N}} \left|\cot\left(\sqrt{k^2 - \mu^{\rm N}_n}(x_{\rm N} - x_{\rm s})\right)\right| \quad \text{and} \quad C_2 := \coth\left(\sqrt{\mu^{\rm N}_{M_{\rm N}+1} - k^2}(x_{\rm N} - x_{\rm s})\right).$$

In fact, assumption (2.26) implies that $C_1 := \cot\left(\sqrt{k^2 - \mu^{\rm N}_{M_{\rm N}}}(x_{\rm N} - x_{\rm s})\right)$.

Obviously,

$$k^2 - \mu^{\rm N}_n < k^2 \le \frac{k^2}{\mu^{\rm N}_1}\, \mu^{\rm N}_n \quad \text{for } n = 1, \dots, M_{\rm N},$$

while $\{\sqrt{\mu_n^{\rm N} - k^2}(x_{\rm N} - x_{\rm s})\}_{n \geq M_{\rm N}+1}$ is an increasing sequence. These remarks and the fact that $\coth(\cdot)$ is a strictly decreasing function in $(0, +\infty)$ allow us to deduce that

$$
\begin{aligned}
|(Ru, v)_{\Gamma_4}| &\leq C_1 k \sum_{n=1}^{M_{\rm N}} |u_n^{\rm N}(x_{\rm N}) v_n^{\rm N}(x_{\rm N})| + C_2 \sum_{n=M_{\rm N}+1}^{\infty} \sqrt{\mu_n^{\rm N}}\, |u_n^{\rm N}(x_{\rm N}) v_n^{\rm N}(x_{\rm N})| \\
&\leq \frac{C_1 k}{\sqrt{\mu_1^{\rm N}}} \sum_{n=1}^{M_{\rm N}} \sqrt{\mu_n^{\rm N}}\, |u_n^{\rm N}(x_{\rm N})|\, |v_n^{\rm N}(x_{\rm N})| + C_2 \sum_{n=M_{\rm N}+1}^{\infty} \sqrt{\mu_n^{\rm N}}\, |u_n^{\rm N}(x_{\rm N})|\, |v_n^{\rm N}(x_{\rm N})| \\
&\leq \max\left\{ C_1 \frac{k}{\sqrt{\mu_1^{\rm N}}}, C_2 \right\} \|u\|_{X^{1/2}(\Gamma_4)} \|v\|_{X^{1/2}(\Gamma_4)} \\
&\leq C_\alpha(k, x_{\rm N})\, \|u\|_{\mathcal{H}}\, \|v\|_{\mathcal{H}}\,,
\end{aligned}
$$

where in the last bound we used a standard trace inequality. The proof of (3.5) is entirely analogous.

For the proof of (3.6) the definition of S, the Cauchy–Schwarz inequality, and the trace inequality on Γ_4 imply

$$
\begin{aligned}
|(S, v)_{\Gamma_4}| &\leq \frac{1}{2} \left| \sum_{n=1}^{M_{\rm N}} \frac{Y_n^{\rm N}(y_{\rm s})}{\sin\left(\sqrt{k^2 - \mu_n^{\rm N}}(x_{\rm N} - x_{\rm s})\right)} \overline{v_n^{\rm N}(x_{\rm N})} \right| \\
&\quad + \frac{1}{2} \left| \sum_{n=M_{\rm N}+1}^{\infty} \frac{Y_n^{\rm N}(y_{\rm s})}{\sinh\left(\sqrt{\mu_n^{\rm N} - k^2}(x_{\rm N} - x_{\rm s})\right)} \overline{v_n^{\rm N}(x_{\rm N})} \right| \\
&\leq \frac{1}{2} \left(\sum_{n=1}^{M_{\rm N}} \frac{(\mu_n^{\rm N})^{-1/2} (Y_n^{\rm N}(y_{\rm s}))^2}{\sin^2\left(\sqrt{k^2 - \mu_n^{\rm N}}(x_{\rm N} - x_{\rm s})\right)} \right)^{1/2} \left(\sum_{n=1}^{M_{\rm N}} (\mu_n^{\rm N})^{1/2}\, |v_n^{\rm N}(x_{\rm N})|^2 \right)^{1/2} \\
&\quad + \frac{1}{2} \left(\sum_{n=M_{\rm N}+1}^{\infty} \frac{(\mu_n^{\rm N})^{-1/2} (Y_n^{\rm N}(y_{\rm s}))^2}{\sinh^2\left(\sqrt{\mu_n^{\rm N} - k^2}(x_{\rm N} - x_{\rm s})\right)} \right)^{1/2} \left(\sum_{n=M_{\rm N}+1}^{\infty} (\mu_n^{\rm N})^{1/2}\, |v_n^{\rm N}(x_{\rm N})|^2 \right)^{1/2} \\
&\leq \|S\|_{X^{-1/2}(\Gamma_4)} \|v\|_{X^{1/2}(\Gamma_4)} \leq C_\gamma\, \|S\|_{X^{-1/2}(\Gamma_4)}\, \|v\|_{\mathcal{H}}\,.
\end{aligned}
$$

□

The form $\mathcal{B}(u, v)$ is sesquilinear, continuous, but, as expected, not positive definite. However, despite the presence of the nonlocal boundary terms, it satisfies a Gårding-type inequality whenever (2.26) holds. Our analysis can be extended irrespectively of the validity of (2.26); see Remark 3.1.

Proposition 3.2. *Assume that x_N satisfies (2.26). Then for all $u \in \mathcal{H}$ there holds*

$$
\operatorname{Re} \mathcal{B}(u, u) \geq \|u\|_{\mathcal{H}}^2 - 2k^2 \|u\|_{\Omega}^2. \tag{3.7}
$$

Proof. Letting $v = u$ in (3.2) and considering separately real and imaginary parts we immediately see that

$$
\operatorname{Re} \mathcal{B}(u, u) = \|\nabla u\|_{\Omega}^2 - k^2 \|u\|_{\Omega}^2 - \operatorname{Re}\,(Tu, u)_{\Gamma_2} + \operatorname{Re}\,(Ru, u)_{\Gamma_4} = -\operatorname{Re}\,(S, u)_{\Gamma_4}. \tag{3.8}
$$

Now, the definitions of R and T (see (2.20)–(2.21), and (2.9)–(2.10), respectively) and the orthonormality of Y_n^{E}'s for E = N or F imply that

$$(R_1u,u)_{\Gamma_4} = \sum_{n=1}^{M_{\mathrm{N}}} \sqrt{k^2-\mu_n^{\mathrm{N}}}\,\cot\Big(\sqrt{k^2-\mu_n^{\mathrm{N}}}(x_{\mathrm{N}}-x_{\mathrm{s}})\Big)\,|u_n^{\mathrm{N}}(x_{\mathrm{N}})|^2, \tag{3.9}$$

$$(R_2u,u)_{\Gamma_4} = \sum_{n=M_{\mathrm{N}}+1}^{\infty} \sqrt{\mu_n^{\mathrm{N}}-k^2}\,\coth\Big(\sqrt{\mu_n^{\mathrm{N}}-k^2}(x_{\mathrm{N}}-x_{\mathrm{s}})\Big)\,|u_n^{\mathrm{N}}(x_{\mathrm{N}})|^2, \tag{3.10}$$

$$(T_1u,u)_{\Gamma_2} = i\sum_{n=1}^{M_{\mathrm{F}}} \sqrt{k^2-\mu_n^{\mathrm{F}}}\,|u_n^{\mathrm{F}}(x_{\mathrm{F}})|^2, \tag{3.11}$$

$$(T_2u,u)_{\Gamma_2} = -\sum_{n=M_{\mathrm{F}}+1}^{\infty} \sqrt{\mu_n^{\mathrm{F}}-k^2}\,|u_n^{\mathrm{F}}(x_{\mathrm{F}})|^2. \tag{3.12}$$

Therefore, (3.8) may be rewritten in the following form:

$$\|\nabla u\|_\Omega^2 - k^2\|u\|_\Omega^2 = (T_2u,u)_{\Gamma_2} - (Ru,u)_{\Gamma_4} - \mathrm{Re}\,(S,u)_{\Gamma_4}. \tag{3.13}$$

Here, we remark that under assumption (2.26) on x_{N}, it is immediately seen from (3.9) that $(R_1u,u)_{\Gamma_4} \geq 0$ and, of course, (3.10) and (3.12) imply that $(R_2u,u)_{\Gamma_4} \geq 0$ and $(T_2u,u)_{\Gamma_2} \leq 0$. Therefore, $\mathrm{Re}\,\mathcal{B}(u,u) = \|\nabla u\|_\Omega^2 - k^2\|u\|_\Omega^2 - (T_2u,u)_{\Gamma_2} + (Ru,u)_{\Gamma_4} \geq \|\nabla u\|_\Omega^2 - k^2\|u\|_\Omega^2$ and the proof is complete. ∎

REMARK 3.1. *Notice that when (2.26) is not assumed to hold there is no guarantee that the term $(R_1u,u)_{\Gamma_4}$ is positive, and thus it should be estimated. This is done in Section 4.2, where the well-posedness analysis is completed without assuming (2.26).*

For future reference we note that

$$\mathrm{Im}\,\mathcal{B}(u,u) = -\mathrm{Im}\,(Tu,u)_{\Gamma_2} = -\mathrm{Im}\,(S,u)_{\Gamma_4} \tag{3.14}$$

and

$$\mathrm{Im}\,(Tu,u)_{\Gamma_2} = \sum_{n=1}^{M_{\mathrm{F}}} \sqrt{k^2-\mu_n^{\mathrm{F}}}\,|u_n^{\mathrm{F}}(x_{\mathrm{F}})|^2 = \mathrm{Im}\,(S,u)_{\Gamma_4}. \tag{3.15}$$

The analysis of the well-posedness is completed in the next section.

4. Well-posedness of the variational problem. We begin the analysis by assuming that the bottom topography of the waveguide is given as the graph of a sufficiently smooth, positive function $y = h(x)$. Let $D_{\max} = \max_{x\in[x_{\mathrm{N}},x_{\mathrm{F}}]} h(x)$, $D_{\min} = \min_{x\in[x_{\mathrm{N}},x_{\mathrm{F}}]} h(x)$, and $L = x_{\mathrm{F}} - x_{\mathrm{N}}$, the distance between the two artificial boundaries. Inspired by the work of Chandler-Wilde and Monk, [9], we introduce the dimensionless wavenumbers

$$\tilde{\kappa} = kD_{\max} \text{ and } \kappa = kL. \tag{4.1}$$

We note that, in general, κ and $\tilde{\kappa}$ should be thought of as being greater than one, since $k = 2\pi/\lambda$, where λ is the wavelength, and in most realistic applications L and the depth of the waveguide support a few wavelengths.

4.1. Existence-uniqueness for a downsloping bottom. Next, we show that under the assumption that the bottom of the waveguide, described by the function h, is increasing, *i.e.*, we are in the case of a *downsloping bottom*, existence and uniqueness are furnished for arbitrarily large wavenumbers. The proof is broken into several steps. The starting point is the use of a test function depending on ∇u; see [23], [24]. In our case we would like to use

$$v = (x - x_{\rm N})u_x.$$

However, due to the boundary conditions, v does not belong to $\mathcal{H}$. One can modify v to belong to $\mathcal{H}$ or, alternatively, do a direct calculation using the Gauss–Green theorem and the fact that u satisfies the Helmholtz equation (3.1). To this end, it will be useful to use the identity

$$2\mathrm{Re}\int_\Omega \Delta u\left(\overline{\boldsymbol{\alpha}\cdot\nabla u}\right) = -\int_{\partial\Omega}\boldsymbol{\alpha}\cdot\boldsymbol{\nu}|\nabla u|^2 + \int_\Omega(\operatorname{div}\boldsymbol{\alpha})|\nabla u|^2 + 2\mathrm{Re}\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\left(\overline{\boldsymbol{\alpha}\cdot\nabla u}\right)$$
$$- 2\mathrm{Re}\int_\Omega\left[\frac{\partial u}{\partial x}\left(\frac{\partial\alpha_1}{\partial x}\overline{\frac{\partial u}{\partial x}} + \frac{\partial\alpha_2}{\partial x}\overline{\frac{\partial u}{\partial y}}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial\alpha_1}{\partial y}\overline{\frac{\partial u}{\partial x}} + \frac{\partial\alpha_2}{\partial y}\overline{\frac{\partial u}{\partial y}}\right)\right], \quad (4.2)$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in (C^1(\overline{\Omega}))^2$ is arbitrary. In our case we use $\boldsymbol{\alpha} = (x - x_{\rm N}, 0)$ in order to have $\boldsymbol{\alpha}\cdot\nabla u = (x - x_{\rm N})u_x$. Identity (4.2) is derived in Cummings and Feng [10, Proposition 1]. The importance of similar identities in the analysis of Helmholtz-type problems can be traced back into the work of Morawetz and Ludwig [28]. Such functions are also used in the analysis of boundary integral methods for Helmholtz problems; see [8] for a comprehensive review. Here it is required that $u \in H^2(\Omega)$, Ω being a star-shaped domain with piecewise smooth boundary, and $\boldsymbol{\nu}$ denotes the outward unit normal vector to $\partial\Omega$.

The first ingredient in our proof is thus the following identity.

LEMMA 4.1. *Assume that $u \in \mathcal{H}$ is a solution of the variational problem (3.2). Then the following identity holds:*

$$2\|u_x\|^2_\Omega = \|\nabla u\|^2_\Omega - k^2\|u\|^2_\Omega + L\Big(\|u_x\|^2_{\Gamma_2} - \|u_y\|^2_{\Gamma_2} + k^2\|u\|^2_{\Gamma_2}\Big)$$
$$- \int_{\Gamma_1}\Big(x - x_N\Big)h'(x)\,|\nabla u|^2\,dx, \quad (4.3)$$

where $L = x_F - x_N$.

Proof. Let us assume that $u \in \mathcal{H}$ is a solution of (3.2). Then it belongs to $H^2(\Omega)$ by standard elliptic regularity results. Thus we apply (4.2) for $\boldsymbol{\alpha} = (x - x_{\rm N}, 0)$, implying $\operatorname{div}\boldsymbol{\alpha} = 1$ and $\boldsymbol{\alpha}\cdot\nabla u = (x - x_{\rm N})u_x$. Note also that since $u = 0$ on Γ_1 and on Γ_3 we conclude that

$$u_x(x,0) = 0, \quad u_x(x,h(x)) + h'(x)\,u_y(x,h(x)) = 0 \text{ for all } x \in [x_{\rm N}, x_{\rm F}]. \quad (4.4)$$

Let us now compute each term in (4.2) separately, keeping in mind the following: On Γ_1, $y = h(x)$ and the outward unit normal $\boldsymbol{\nu} = (-h'(x), 1)/\sqrt{1 + (h'(x))^2}$; on Γ_2, $x = x_{\rm F}$ and $\boldsymbol{\nu} = (1, 0)$; on Γ_3, $y = 0$ and $\boldsymbol{\nu} = (0, -1)$; on Γ_4, $x = x_{\rm N}$ and $\boldsymbol{\nu} = (-1, 0)$.

For the first term in the right hand side of (4.2) we have

$$\int_{\partial\Omega}\boldsymbol{\alpha}\cdot\boldsymbol{\nu}|\nabla u|^2 = \int_{\Gamma_1}\Big(x - x_{\rm N}, 0\Big)\cdot\frac{(-h'(x), 1)}{\sqrt{1 + (h'(x))^2}}\,|\nabla u|^2\sqrt{1 + (h'(x))^2}\,dx$$
$$+ \int_{\Gamma_2}(x_{\rm F} - x_{\rm N}, 0)\cdot(1, 0)\,|\nabla u|^2 dy$$
$$+ \int_{\Gamma_3}(x - x_{\rm N}, 0)\cdot(0, -1)\,|\nabla u|^2 dx + \int_{\Gamma_4}(x_{\rm N} - x_{\rm N}, 0)\cdot(-1, 0)\,|\nabla u|^2 dy.$$

Therefore,

$$\int_{\partial\Omega} \boldsymbol{\alpha}\cdot\boldsymbol{\nu}|\nabla u|^2 = -\int_{\Gamma_1}\Big(x - x_{\rm N}\Big)h'(x)\,|\nabla u|^2\,dx + (x_{\rm F}-x_{\rm N})\int_{\Gamma_2}|\nabla u|^2 dy. \tag{4.5}$$

The second term in the right hand side of (4.2) is just $\int_\Omega |\nabla u|^2$.

For the next term, notice

$$\begin{aligned}
&2\mathrm{Re}\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\left(\overline{\boldsymbol{\alpha}\cdot\nabla u}\right) = \\
&= 2\mathrm{Re}\int_{\Gamma_1}\left(\nabla u\cdot\frac{(-h'(x),1)}{\sqrt{1+(h'(x))^2}}\right)(x-x_{\rm N})\bar u_x\,\sqrt{1+(h'(x))^2}\,dx \\
&\ +2\mathrm{Re}\int_{\Gamma_2}(x_{\rm F}-x_{\rm N})u_x\bar u_x\,dy - 2\mathrm{Re}\int_{\Gamma_3}(x-x_{\rm N})u_y\bar u_x\,dx - 2\mathrm{Re}\int_{\Gamma_4}(x_{\rm N}-x_{\rm N})u_x\bar u_x\,dy \\
&= -2\mathrm{Re}\int_{\Gamma_1}(x-x_{\rm N})\Big(-h'(x)|u_x|^2-\bar u_x u_y\Big)\,dx + 2(x_{\rm F}-x_{\rm N})\int_{\Gamma_2}|u_x|^2\,dy.
\end{aligned}$$

Note that (4.4) implies that $\bar u_x + h'(x)\bar u_y = 0$; therefore $\bar u_x u_y = -h'(x)\,|u_y|^2$. Thus,

$$2\mathrm{Re}\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\left(\overline{\boldsymbol{\alpha}\cdot\nabla u}\right) = -2\int_{\Gamma_1}(x-x_{\rm N})h'(x)\,|\nabla u|^2\,dx + 2(x_{\rm F}-x_{\rm N})\int_{\Gamma_2}|u_x|^2\,dy. \tag{4.6}$$

Finally,

$$2\mathrm{Re}\int_\Omega\left[\frac{\partial u}{\partial x}\left(\frac{\partial\alpha_1}{\partial x}\overline{\frac{\partial u}{\partial x}}+\frac{\partial\alpha_2}{\partial x}\overline{\frac{\partial u}{\partial y}}\right)+\frac{\partial u}{\partial y}\left(\frac{\partial\alpha_1}{\partial y}\overline{\frac{\partial u}{\partial x}}+\frac{\partial\alpha_2}{\partial y}\overline{\frac{\partial u}{\partial y}}\right)\right] = 2\int_\Omega |u_x|^2.$$

Since $2\mathrm{Re}\int_\Omega \Delta u\left(\overline{\boldsymbol{\alpha}\cdot\nabla u}\right) = 2\mathrm{Re}\int_\Omega (x-x_{\rm N})\Delta u\,\bar u_x$ we therefore conclude

$$\begin{aligned}
2\mathrm{Re}\int_\Omega (x-x_{\rm N})\Delta u\,\bar u_x &= \int_{\Gamma_1}(x-x_{\rm N})h'(x)\,|\nabla u|^2\,dx - L\int_{\Gamma_2}|\nabla u|^2 dy \\
&\quad + \int_\Omega |\nabla u|^2 - 2\int_{\Gamma_1}(x-x_{\rm N})h'(x)\,|\nabla u|^2\,dx + 2L\int_{\Gamma_2}|u_x|^2\,dy - 2\int_\Omega |u_x|^2.
\end{aligned}$$

Since $-\Delta u = k^2 u$ in $L^2(\Omega)$ we have that

$$\begin{aligned}
-2\,k^2\mathrm{Re}\int_\Omega (x-x_{\rm N})u\,\bar u_x = -\int_{\Gamma_1}(x-x_{\rm N})h'(x)\,|\nabla u|^2\,dx - L\|u_y\|^2_{\Gamma_2} + L\|u_x\|^2_{\Gamma_2} \\
+\|\nabla u\|^2_\Omega - 2\|u_x\|^2_\Omega.
\end{aligned}$$

Note that the first integral in the equation above may be written as

$$-2\,\mathrm{Re}\int_\Omega (x-x_{\rm N})u\,\bar u_x = -\int_\Omega (x-x_{\rm N})(|u|^2)_x = \int_\Omega |u|^2 - L\int_{\Gamma_2}|u|^2,$$

and the proof is complete. □

The following bound shows that $\|u\|^2_\Omega$ can be controlled by $\|u_x\|^2_\Omega$ and boundary terms.

LEMMA 4.2. *For all $u \in \mathcal{H}$*

$$\|u\|_{\Omega}^2 \leq 2L\|u\|_{\Gamma_2}^2 + L^2\|u_x\|_{\Omega}^2. \tag{4.7}$$

Proof. We first consider u smooth with $u = 0$ on $\Gamma_1 \cup \Gamma_3$. Since the function h describing the bottom curve Γ_1 is increasing we are allowed to write

$$u(x, y) = u(x_{\mathrm{F}}, y) - \int_x^{x_{\mathrm{F}}} \frac{\partial}{\partial x} u(s, y)\, ds.$$

Thus

$$\int_{x_{\mathrm{N}}}^{x_{\mathrm{F}}} |u(s, y)|^2 ds \leq 2L|u(x_{\mathrm{F}}, y)|^2 + 2\int_{x_{\mathrm{N}}}^{x_{\mathrm{F}}} (x_{\mathrm{F}} - x) dx \int_{x_{\mathrm{N}}}^{x_{\mathrm{F}}} |u_x(s, y)|^2\, ds,$$

i.e.,

$$\int_\Omega |u(x, y)|^2 \leq 2L\int_0^{D_{\mathrm{F}}} |u(x_{\mathrm{F}}, y)|^2\, dy + L^2\int_\Omega |u_x(x, y)|^2, \tag{4.8}$$

and the proof follows by a density argument. □

The following lemma shows that it is possible to control the terms which appear in the parentheses in the right-hand side (r.h.s.) of (4.3). It is inspired by an analogous result in [9, Lemma 2.2].

LEMMA 4.3. *If $u \in \mathcal{H}$ is a solution of the variational problem (3.2) then*

$$\|u_x\|_{\Gamma_2}^2 - \|u_y\|_{\Gamma_2}^2 + k^2\|u\|_{\Gamma_2}^2 \leq 2k\,\mathrm{Im}\,(S, u)_{\Gamma_4}. \tag{4.9}$$

Proof. As mentioned, $X^1(\Gamma_2) = \overset{0}{H}{}^1(\Gamma_2)$; thus

$$\|u_y\|_{\Gamma_2}^2 = \|u\|_{X^1(\Gamma_2)}^2 = \sum_{n=1}^{\infty} \mu_n^{\mathrm{F}} |u_n^{\mathrm{F}}(x_{\mathrm{F}})|^2.$$

See also [5].

Moreover, the orthonormality of the eigenfunctions Y_m^{F} and (2.8)–(2.10) imply that

$$\|u_x\|_{\Gamma_2}^2 = \|Tu\|_{\Gamma_2}^2 = \sum_{n=1}^{M_F} (k^2 - \mu_n^{\mathrm{F}})\, |u_n^{\mathrm{F}}(x_{\mathrm{F}})|^2 + \sum_{n=M_F+1}^{\infty} (\mu_n^{\mathrm{F}} - k^2)\, |u_n^{\mathrm{F}}(x_{\mathrm{F}})|^2.$$

In summary,

$$\begin{aligned} &\|u_x\|_{\Gamma_2}^2 - \|u_y\|_{\Gamma_2}^2 + k^2\|u\|_{\Gamma_2}^2 = \\ &= 2\sum_{n=1}^{M_F} (k^2 - \mu_n^{\mathrm{F}})\, |u_n^{\mathrm{F}}(x_{\mathrm{F}})|^2 \leq 2k\sum_{n=1}^{M_F} \sqrt{k^2 - \mu_n^{\mathrm{F}}}\, |u_n^{\mathrm{F}}(x_{\mathrm{F}})|^2 = 2k\,\mathrm{Im}\,(Tu, u)_{\Gamma_2}. \end{aligned}$$

The result now follows from (3.15). □

We are now in a position to establish an *a priori* bound for the solutions of (3.2).

THEOREM 4.4. *If $u \in \mathcal{H}$ is a solution of the variational problem (3.2), h is an increasing smooth function, x_N is such that (2.25) holds, and x_F is chosen so that $L = x_F - x_N$ is large enough, then*

$$\|u\|_{\mathcal{H}} \le C(\kappa)\, \|S\|_{X^{-1/2}(\Gamma_4)}, \tag{4.10}$$

where

$$C(\kappa) = C_\gamma \left(1 + \frac{4\kappa}{\sqrt{1 - \mu^{F}_{M_F}/k^2}} + \kappa^2 + 2\kappa^3 \right)$$

with C_γ the constant in (3.6).

Proof. We start by noting that (3.13) implies

$$\|u\|^2_{\mathcal{H}} = 2k^2 \|u\|^2_\Omega + (T_2 u, u)_{\Gamma_2} - (Ru, u)_{\Gamma_4} - \operatorname{Re}(S, u)_{\Gamma_4}.$$

Since $(T_2 u, u)_{\Gamma_2} \le 0$, and (2.26) guarantees that $(Ru, u)_{\Gamma_4} \ge 0$, we deduce that

$$\|u\|^2_{\mathcal{H}} \le 2k^2 \|u\|^2_\Omega - \operatorname{Re}(S, u)_{\Gamma_4}. \tag{4.11}$$

Now, (4.3) using (4.9) and (3.13) and recalling that $\kappa = kL$, imply that

$$\begin{aligned} 2\|u_x\|^2_\Omega \le{}& (T_2 u, u)_{\Gamma_2} - (Ru, u)_{\Gamma_4} - \operatorname{Re}(S, u)_{\Gamma_4} + 2\kappa \operatorname{Im}(S, u)_{\Gamma_4} \\ &- \int_{\Gamma_1} (x - x_{\mathrm{N}}) h'(x)\, |\nabla u|^2 \, dx. \end{aligned} \tag{4.12}$$

Since $(Ru, u)_{\Gamma_4} \ge 0$ and h is an increasing function we deduce that the second and fifth terms in the r.h.s. of (4.12) are nonpositive. Thus

$$2\|u_x\|^2_\Omega \le (T_2 u, u)_{\Gamma_2} - \operatorname{Re}(S, u)_{\Gamma_4} + 2\kappa \operatorname{Im}(S, u)_{\Gamma_4}.$$

Multiplying (4.7) by $2k^2$, and combining the resulting inequality with the one above, we arrive at

$$2k^2 \|u\|^2_\Omega \le 4k\kappa \|u\|^2_{\Gamma_2} + \kappa^2 (T_2 u, u)_{\Gamma_2} - \kappa^2 \operatorname{Re}(S, u)_{\Gamma_4} + 2\kappa^3 \operatorname{Im}(S, u)_{\Gamma_4},$$

which along with (4.11) shows that

$$\|u\|^2_{\mathcal{H}} \le 4k\kappa \|u\|^2_{\Gamma_2} + \kappa^2\, (T_2 u, u)_{\Gamma_2} - (1 + \kappa^2) \operatorname{Re}(S, u)_{\Gamma_4} + 2\kappa^3 \operatorname{Im}(S, u)_{\Gamma_4}.$$

To complete the proof we need to control the first two terms of the r.h.s. in the previous inequality. Indeed,

$$\begin{aligned} 4k\, \|u\|^2_{\Gamma_2} + \kappa\, (T_2 u, u)_{\Gamma_2} &= 4k\, \|u\|^2_{X^0(\Gamma_2)} + \kappa\, (T_2 u, u)_{\Gamma_2} \\ &= 4k \sum_{n=1}^{M_{\mathrm{F}}} |u^{\mathrm{F}}_n(x_{\mathrm{F}})|^2 + k \sum_{n=M_{\mathrm{F}}+1}^{\infty} \left(4 - L\sqrt{\mu^{\mathrm{F}}_n - k^2} \right) |u^{\mathrm{F}}_n(x_{\mathrm{F}})|^2. \end{aligned}$$

Since $\{\mu^{\mathrm{F}}_n\}$ is an increasing sequence it is enough to choose an x_{F} such that

$$4 - L\sqrt{\mu^{\mathrm{F}}_{M_{\mathrm{F}}+1} - k^2} \le 0, \tag{4.13}$$

thus ensuring $4 - L\sqrt{\mu_n^{\mathrm{F}} - k^2} \le 0$ for all $n \ge M_{\mathrm{F}} + 1$. Then we conclude that

$$4k\,\|u\|_{\Gamma_2}^2 + \kappa\,(T_2u,u)_{\Gamma_2} \le 4k\sum_{n=1}^{M_{\mathrm{F}}}|u_n^{\mathrm{F}}(x_{\mathrm{F}})|^2 \le \frac{4k}{\sqrt{k^2-\mu_{M_{\mathrm{F}}}^{\mathrm{F}}}}\sum_{n=1}^{M_{\mathrm{F}}}\sqrt{k^2-\mu_n^{\mathrm{F}}}\,|u_n^{\mathrm{F}}(x_{\mathrm{F}})|^2$$
$$= \frac{4}{\left(1-\mu_{M_{\mathrm{F}}}^{\mathrm{F}}/k^2\right)^{1/2}}\operatorname{Im}(S,u)_{\Gamma_4}, \tag{4.14}$$

where the second inequality holds since $\sqrt{k^2-\mu_1^{\mathrm{F}}} \ge \cdots \ge \sqrt{k^2-\mu_{M_{\mathrm{F}}}^{\mathrm{F}}}$, and the last equality comes from (3.15).

Therefore, we end up with

$$\|u\|_{\mathcal{H}}^2 \le \left(1+\frac{4\kappa}{\left(1-\mu_{M_{\mathrm{F}}}^{\mathrm{F}}/k^2\right)^{1/2}}+\kappa^2+2\kappa^3\right)\,|(S,u)_{\Gamma_4}|,$$

and the proof is completed using (3.6). ∎

4.1.1. Existence. The *a priori* bound in Theorem 4.4 implies uniqueness. It turns out that it implies existence as well. Indeed, motivated by [7] we state the following application of Banach's closed range theorem.

LEMMA 4.5. *Let H be a Hilbert space and H^* its dual. Assume that $\mathcal{B}(\cdot,\cdot)$ is a continuous sesquilinear form on $H\times H$ and, further, whenever solutions $u,w\in H$ of*

$$\mathcal{B}(u,v) = \overline{G_1(v)}, \quad v\in H, \tag{4.15}$$

and

$$\mathcal{B}(v,w) = G_2(v), \quad v\in H, \tag{4.16}$$

exist they satisfy the a priori bounds

$$\|u\|_H \le c_1\|G_1\|_{H^*} \quad \textit{and} \quad \|w\|_H \le c_2\|G_2\|_{H^*}, \tag{4.17}$$

where $G_1, G_2 \in H^*$, and c_1, c_2 are positive constants. Then for each $G_1\in H^*$ there exists a solution of problem (4.15).

Proof. Let $\overline{H}^*$ be the space of antilinear (conjugate linear) functionals on H. We define the operator $A: H\to \overline{H}^*$ as

$$Au(v) := \mathcal{B}(u,v), \quad v\in H. \tag{4.18}$$

Since $\mathcal{B}(u,v)$ is continuous, A is linear in u and well defined. If u is a solution of (4.15), then $Au = \overline{G_1}$. The first a priori bound in (4.17) can be written as

$$\|u\|_H \le c_1\|\overline{Au}\|_{H^*}\,;$$

hence the range of A, denoted by $\mathcal{R}(A)$, is closed and A is 1-1. It suffices to show that A is onto $\overline{H}^*$. Banach's closed range theorem [33, Theorem VII.5], [6, Theorem 2.19] implies that if A' is the dual of A, and $\mathcal{N}(A')$ its null space, then $\mathcal{R}(A) = \mathcal{N}(A')^{\perp}$. But the second a priori bound in (4.17) implies that A' is 1-1; thus A is onto. ∎

REMARK 4.1. *The assumption on the dual problem can be relaxed to assume just uniqueness for (4.16). The proof is valid for the more general case where $\mathcal{B}(\cdot,\cdot)$ is defined on $X\times Y$, with*

X, Y reflexive Banach spaces. The above result has many different statements and its proof is well known [29], [2]. It is essentially equivalent to Babuška's Theorem 2.1 [2] based on inf-sup type of assumptions on the bilinear form $\mathcal{B}(u,v)$. In the proof of [2, Theorem 2.1] the inf-sup assumptions are implicitly transformed into bounds of the form (4.17); therefore, in our case, it is preferable to use the statement of Lemma 3.1, since we establish (4.17) directly.

The following theorem completes the analysis of the well-posedness.

THEOREM 4.6. *There exists a unique solution of the problem (3.2) which satisfies the a priori bound (4.10).*

Proof. For $v \in \mathcal{H}$, the problem (3.2) is written in the form (4.15), where the sesquilinear form $\mathcal{B}(u,v)$ is defined in (3.3) and $G_1(v) = -\overline{(S,v)}_{\Gamma_4}$. From (3.6) we have that G_1 is a bounded linear functional on $\mathcal{H}$. It is easy to verify that $\mathcal{B}(u,v) = \mathcal{B}(\overline{v},\overline{u})$. Then $G_2(v) = \overline{G_1(\overline{v})}$ in (4.16), and the estimates (4.17) follow immediately from (4.10). Hence, Lemma 4.5 can be applied to derive existence. The proof follows in view of Theorem 4.4. ∎

4.2. Relaxing the assumption (2.25) to (2.24). Motivated by the numerical experiments, we observe that the validity of (2.25), namely, $\cot\left(\sqrt{k^2-\mu_n^{\mathrm{N}}}(x_{\mathrm{N}}-x_{\mathrm{s}})\right) > 0$, for $n = 1,\dots,M_{\mathrm{N}}$, seems to play no role in the behavior of the approximate solutions. Therefore, it seems reasonable to establish well-posedeness even when (2.25) is not valid. Indeed, in this section we describe how one can actually remove the assumption (2.25) and require only x_{N} to satisfy the much milder condition (2.24), *i.e.*, $\sin\left(\sqrt{k^2-\mu_n^{\mathrm{N}}}(x_{\mathrm{N}}-x_{\mathrm{s}})\right) \neq 0$, for $n = 1,\dots,M_{\mathrm{N}}$. Assumption (2.25) is needed in the proof of Theorem 4.4 to obtain (4.11) by forcing the term

$$(R_1u,u)_{\Gamma_4} = \sum_{n=1}^{M_{\mathrm{N}}} \sqrt{k^2-\mu_n^{\mathrm{N}}}\,\cot\Big(\sqrt{k^2-\mu_n^{\mathrm{N}}}(x_{\mathrm{N}}-x_{\mathrm{s}})\Big)\,|u_n^{\mathrm{N}}(x_{\mathrm{N}})|^2$$

to be positive, see Remark 3.1. Here, let us assume that x_{N} is chosen without taking care of the sign of $\cot\Big(\sqrt{k^2-\mu_n^{\mathrm{N}}}(x_{\mathrm{N}}-x_{\mathrm{s}})\Big)$ for $n = 1,\dots,M_{\mathrm{N}}$ and denote

$$-C_* = \min_{n=1,\dots,M_{\mathrm{N}}} \cot\Big(\sqrt{k^2-\mu_n^{\mathrm{N}}}(x_{\mathrm{N}}-x_{\mathrm{s}})\Big),$$

which we assume is negative, *i.e.*, $C_* \geq 0$. Then, of course,

$$(R_1u,u)_{\Gamma_4} \geq -C_* \sum_{n=1}^{M_{\mathrm{N}}} \sqrt{k^2-\mu_n^{\mathrm{N}}}\,|u_n^{\mathrm{N}}(x_{\mathrm{N}})|^2,$$

hence

$$\begin{aligned} -\ (\ R_1u,u)_{\Gamma_4} &\leq C_* \sum_{n=1}^{M_{\mathrm{N}}} \sqrt{k^2-\mu_n^{\mathrm{N}}}\,|u_n^{\mathrm{N}}(x_{\mathrm{N}})|^2 \\ &\leq C_*\Big(\sum_{n=1}^{M_{\mathrm{N}}} |u_n^{\mathrm{N}}(x_{\mathrm{N}})|^2\Big)^{1/2}\Big(\sum_{n=1}^{M_{\mathrm{N}}}(k^2-\mu_n^{\mathrm{N}})|u_n^{\mathrm{N}}(x_{\mathrm{N}})|^2\Big)^{1/2} \\ &\leq C_*\|u\|_{\Gamma_4}\Big(\sum_{n=1}^{M_{\mathrm{N}}}(k^2-\mu_n^{\mathrm{N}})|u_n^{\mathrm{N}}(x_{\mathrm{N}})|^2\Big)^{1/2}. \end{aligned} \tag{4.19}$$

One of the key points in this section is the following lemma, which provides an identity relating norms of the solution of the variational problem (3.2), and of its derivatives, on the various parts of $\partial\Omega$.

LEMMA 4.7. *Assume that $u \in \mathcal{H}$ is a solution of the variational problem (3.2). Then the following identity holds:*

$$\|u_x\|_{\Gamma_4}^2 - \|u_y\|_{\Gamma_4}^2 + k^2\|u\|_{\Gamma_4}^2 = \|u_x\|_{\Gamma_2}^2 - \|u_y\|_{\Gamma_2}^2 + k^2\|u\|_{\Gamma_2}^2 - \int_{\Gamma_1} h'(x)\,|\nabla u|^2\,dx.$$

Proof. The desired identity follows if we apply Rellich's identity (4.2) for $\boldsymbol{\alpha} = (1,0)$. □

Now, for a downsloping bottom, where $h'(x) \geq 0$, Lemma 4.7 implies that

$$\|u_x\|_{\Gamma_4}^2 + k^2\|u\|_{\Gamma_4}^2 - \|u_y\|_{\Gamma_4}^2 \leq \|u_x\|_{\Gamma_2}^2 - \|u_y\|_{\Gamma_2}^2 + k^2\|u\|_{\Gamma_2}^2,$$

and using (4.9) and the fact that $\|u_y\|_{\Gamma_4}^2 = \|u\|_{X^1(\Gamma_4)}^2$, we arrive at

$$\|u_x\|_{\Gamma_4}^2 + k^2\|u\|_{\Gamma_4}^2 - \|u\|_{X^1(\Gamma_4)}^2 \leq 2k\,\mathrm{Im}\,(S,u)_{\Gamma_4}.$$

Recalling that $u_x = Ru + S$ on Γ_4, we have

$$\|u_x\|_{\Gamma_4}^2 = (Ru+S, Ru+S)_{\Gamma_4} \geq \|Ru\|_{\Gamma_4}^2 - 2|(Ru,S)_{\Gamma_4}|.$$

Therefore

$$\begin{aligned}\|Ru\|_{\Gamma_4}^2 - 2|(Ru,S)_{\Gamma_4}| + k^2\|u\|_{\Gamma_4}^2 - \|u\|_{X^1(\Gamma_4)}^2 &\leq \|u_x\|_{\Gamma_4}^2 + k^2\|u\|_{\Gamma_4}^2 - \|u\|_{X^1(\Gamma_4)}^2\\ &\leq 2k\,\mathrm{Im}\,(S,u)_{\Gamma_4},\end{aligned}$$

i.e.,

$$\|Ru\|_{\Gamma_4}^2 + k^2\|u\|_{\Gamma_4}^2 - \|u\|_{X^1(\Gamma_4)}^2 \leq 2k\,\mathrm{Im}\,(S,u)_{\Gamma_4} + 2|(Ru,S)_{\Gamma_4}|.$$

The above may be written as

$$\begin{aligned}&\sum_{n=1}^{M_{\mathrm{N}}}(k^2-\mu_n^{\mathrm{N}})\cot^2\Big(\sqrt{k^2-\mu_n^{\mathrm{N}}}(x_{\mathrm{N}}-x_{\mathrm{s}})\Big)|v_n^{\mathrm{N}}(x_{\mathrm{N}})|^2\\
&+\sum_{n=M_{\mathrm{N}}+1}^{\infty}(\mu_n^{\mathrm{N}}-k^2)\coth^2\Big(\sqrt{\mu_n^{\mathrm{N}}-k^2}(x_{\mathrm{N}}-x_{\mathrm{s}})\Big)|v_n^{\mathrm{N}}(x_{\mathrm{N}})|^2\\
&+\sum_{n=1}^{M_{\mathrm{N}}}(k^2-\mu_n^{\mathrm{N}})\,|u_n^{\mathrm{N}}(x_{\mathrm{N}})|^2-\sum_{n=M_{\mathrm{N}}+1}^{\infty}(\mu_n^{\mathrm{N}}-k^2)|v_n^{\mathrm{N}}(x_{\mathrm{N}})|^2=\\
&=\sum_{n=1}^{M_{\mathrm{N}}}(k^2-\mu_n^{\mathrm{N}})\left[\cot^2\Big(\sqrt{k^2-\mu_n^{\mathrm{N}}}(x_{\mathrm{N}}-x_{\mathrm{s}})\Big)+1\right]|v_n^{\mathrm{N}}(x_{\mathrm{N}})|^2\\
&+\sum_{n=M_{\mathrm{N}}+1}^{\infty}(\mu_n^{\mathrm{N}}-k^2)\left[\coth^2\Big(\sqrt{\mu_n^{\mathrm{N}}-k^2}(x_{\mathrm{N}}-x_{\mathrm{s}})\Big)-1\right]|v_n^{\mathrm{N}}(x_{\mathrm{N}})|^2\\
&\leq 2k\,\mathrm{Im}\,(S,u)_{\Gamma_4}+2|(Ru,S)_{\Gamma_4}|.\end{aligned}$$

Since $\coth^2\left(\sqrt{\mu_n^{\rm N}-k^2}(x_{\rm N}-x_{\rm S})\right)-1\geq 0$, for $n\geq M_{\rm N}+1$, the inequality above implies that

$$\sum_{n=1}^{M_{\rm N}}(k^2-\mu_n^{\rm N})|u_n^{\rm N}(x_{\rm N})|^2\leq 2k\,|(S,u)_{\Gamma_4}|+2|(Ru,S)_{\Gamma_4}|$$
$$\leq 2\left(\|Ru\|_{X^{1/2}(\Gamma_4)}+k\|u\|_{X^{1/2}(\Gamma_4)}\right)\|S\|_{X^{-1/2}(\Gamma_4)}.$$

The above inequality and (4.19) imply that

$$-(R_1u,u)_{\Gamma_4}\leq\sqrt{2}\,C_*\|u\|_{\Gamma_4}\left(\|Ru\|_{X^{1/2}(\Gamma_4)}+k\|u\|_{X^{1/2}(\Gamma_4)}\right)^{1/2}\|S\|^{1/2}_{X^{-1/2}(\Gamma_4)}.$$

Hence, applying (twice) the arithmetic-geometric mean inequality we get

$$\begin{aligned}-(R_1u,u)_{\Gamma_4}&\leq 2^{-1/2}C_*\|u\|_{\Gamma_4}\left[\varepsilon_1\left(\|Ru\|_{X^{1/2}(\Gamma_4)}+k\|u\|_{X^{1/2}(\Gamma_4)}\right)+\varepsilon_1^{-1}\|S\|_{X^{-1/2}(\Gamma_4)}\right]\\&\leq 2^{-1/2}C_*\left(\varepsilon_1\|u\|_{\Gamma_4}\|Ru\|_{X^{1/2}(\Gamma_4)}+\varepsilon_1k\|u\|_{\Gamma_4}\|u\|_{X^{1/2}(\Gamma_4)}+\frac{\varepsilon_2}{\varepsilon_1}\|u\|^2_{\Gamma_4}\right.\\&\qquad\left.+\frac{1}{4\varepsilon_1\varepsilon_2}\|S\|^2_{X^{-1/2}(\Gamma_4)}\right).\end{aligned}\tag{4.20}$$

If $u\in\mathcal{H}$ is a solution of the variational problem, then it is easily seen that $u\in H^2(\Omega)\cap\mathcal{H}$. Then standard elliptic regularity estimates imply

$$\|u\|_{2,\Omega}\leq C\|\Delta u\|\,.$$

Therefore, R may be viewed as a bounded linear operator from $X^{3/2}(\Gamma_4)$ into $X^{1/2}(\Gamma_4)$, and

$$\|Ru\|_{X^{1/2}(\Gamma_4)}\leq C\|u\|_{X^{3/2}(\Gamma_4)}\leq C\|u\|_{2,\Omega}\leq C\,\|u\|_{\mathcal{H}}\,.\tag{4.21}$$

Next, (4.20), the standard trace inequalities for functions in $L^2(\Gamma_4)$ and in $X^{1/2}(\Gamma_4)$, and (4.21) imply that there exist positive constants C_1,C_2,C_3, depending on k, such that

$$-(R_1u,u)_{\Gamma_4}\leq C_1\varepsilon_1\,\|u\|^2_{\mathcal{H}}+C_2\varepsilon_1^{-1}\varepsilon_2\,\|u\|^2_{\mathcal{H}}+C_3(\varepsilon_1\varepsilon_2)^{-1}\|S\|^2_{X^{-1/2}(\Gamma_4)}.\tag{4.22}$$

We are now in a position to prove the following proposition.

PROPOSITION 4.8. *If $u\in\mathcal{H}$ is a solution to the variational problem (3.2), h is an increasing smooth function, and $x_{\rm N}$ is such that (2.24) holds, then there exists a constant $C=C(\kappa)$ such that*

$$\|u\|_{\mathcal{H}}\leq C(\kappa)\|S\|_{X^{-1/2}(\Gamma_4)}.$$

Proof. We follow the steps of the proof of Theorem 4.4 and describe all the necessary modifications. We begin by noting that in the case where $(R_1u,u)_{\Gamma_4}$ is not assumed to be nonnegative, (4.11) takes the form

$$\|u\|^2_{\mathcal{H}}\leq 2k^2\|u\|^2_{\Omega}-(R_1u,u)_{\Gamma_4}-\mathrm{Re}\,(S,u)_{\Gamma_4},\tag{4.23}$$

while (4.12) implies

$$2\|u_x\|_\Omega^2 \le (T_2u,u)_{\Gamma_2} - (R_1u,u)_{\Gamma_4} - \mathrm{Re}\,(S,u)_{\Gamma_4} + 2\kappa\,\mathrm{Im}\,(S,u)_{\Gamma_4}.$$

Muliplying (4.7) by $2k^2$ and replacing the last term in the resulting inequality with the aid of the above inequality we conclude that

$$2k^2\|u\|_\Omega^2 \le 4k\kappa\|u\|_{\Gamma_2}^2 + \kappa^2(T_2u,u)_{\Gamma_2} - \kappa^2(R_1u,u)_{\Gamma_4} - \kappa^2\mathrm{Re}\,(S,u)_{\Gamma_4} + 2\kappa^3\,\mathrm{Im}\,(S,u)_{\Gamma_4}.$$

Inequality (4.23), the inequality above, (4.14), and the arithmetic-geometric mean inequality imply that

$$\|u\|_{\mathcal{H}}^2 \le C|(S,u)_{\Gamma_4}| - (1+\kappa^2)\,(R_1u,u)_{\Gamma_4} \le C\|S\|_{X^{-1/2}(\Gamma_4)}\,\|u\|_{\mathcal{H}} - (1+\kappa^2)\,(R_1u,u)_{\Gamma_4}. \quad (4.24)$$

The proof is therefore completed using (4.22) for suitable choices of $\varepsilon_1, \varepsilon_2$. □

4.3. Existence-uniqueness for small frequency. Following along the lines of [9, section 3] it is possible to derive existence and uniqueness of the problem (3.2) for an arbitrary bottom profile under the assumption that $kD_{\max}$ is sufficiently small. This may be viewed as a small-frequency/shallow-water assumption, where this terminology is borrowed from [11].

We shall need the following Lemma.

LEMMA 4.9. *For all $u \in \mathcal{H}$ the following Poincaré-type inequality holds:*

$$\|u\|_\Omega \le D_{\max}\|u_y\|_\Omega. \quad (4.25)$$

Proof. Let u be a smooth function with $u = 0$ on $\Gamma_1 \cup \Gamma_3$. It is convenient to use the change of variables

$$x = r, \quad y = z\,h(r), \quad u(x,y) = w(r,z), \quad (4.26)$$

which maps the domain Ω onto the rectangle $\widetilde{\Omega} := \{(r,z) :\ r_{\mathrm{N}} \le r \le r_{\mathrm{F}},\ 0 \le z \le 1\}$, where $r_{\mathrm{N}} = x_{\mathrm{N}}$ and $r_{\mathrm{F}} = x_{\mathrm{F}}$; see also [30, 11].

Then, w is defined on $\widetilde{\Omega}$, $w(r,0) = w(r,1) = 0$, and since $w(r,z) = \int_0^z \frac{\partial}{\partial z} w(r,s)\,ds$ we have

$$\int_{\widetilde{\Omega}} |w(r,z)|^2 dzdr \le \int_{\widetilde{\Omega}} |w_z(r,z)|^2 dzdr.$$

Returning to the original variables shows that $\int_\Omega \frac{1}{h(x)}|u|^2 \le \int_\Omega h(x)|u_y|^2$ and (4.25) follows by density and the fact that $h(x) \le D_{\max}$ for all $x \in [x_{\mathrm{N}}, x_{\mathrm{F}}]$. □

We prove the next lemma, which is analogous to [9, Lemma 3.6].

LEMMA 4.10. *For all $u \in \mathcal{H}$, and under assumption (2.25),*

$$\mathrm{Re}\,\mathcal{B}(u,u) \ge \frac{1-\tilde{\kappa}^2}{1+\tilde{\kappa}^2}\,\|u\|_{\mathcal{H}}^2\,.$$

Proof. By (4.25) and the first relation in (4.1) we see that $\|u_y\|_\Omega^2 \ge \frac{k^2}{\tilde{\kappa}^2}\|u\|_\Omega^2$, and using this, one may easily verify that $\|u\|_{\mathcal{H}}^2 \ge \|u_y\|_\Omega^2 + k^2\|u\|_\Omega^2 \ge k^2\left(\frac{1+\tilde{\kappa}^2}{\tilde{\kappa}^2}\right)\|u\|_\Omega^2$. Now (3.7) and the last relation imply that $\mathrm{Re}\,\mathcal{B}(u,u) \ge \|u\|_{\mathcal{H}}^2 - 2k^2\|u\|_\Omega^2 \ge \frac{1-\tilde{\kappa}^2}{1+\tilde{\kappa}^2}\,\|u\|_{\mathcal{H}}^2\,.$ □

Lemma 4.10, the boundedness of the sesquilinear form (discussed in §3), and the Lax–Milgram lemma guarantee the existence of a unique solution of (3.2), under the assumption $\tilde{\kappa} < 1$, *i.e.*, $kD_{\max} < 1$.

In our case, where we study acoustic wave propagation in a waveguide, in contrast to the problem studied in [9], this assumption is very restrictive, so the result above is of small practical importance. Indeed, recalling that the number of modes which propagate in the deepest part of the waveguide is $\lfloor \frac{2D_{\max}}{\lambda} \rfloor$ and $\lambda = 2\pi/k$, we see that

$$\left\lfloor \frac{2D_{\max}}{\lambda} \right\rfloor = \left\lfloor \frac{kD_{\max}}{\pi} \right\rfloor = 0,$$

since $kD_{\max} < 1$. Stated otherwise, the wavenumbers which are allowed by the constraint $kD_{\max} < 1$ correspond to frequencies which are below the *cutoff frequency*, (cf. [19, §2.4.4.4]), *i.e.*, the field is evanescent in the whole waveguide.

5. Numerical experiments. In this section we include numerical experiments which support the claim that the model suggested herein indeed provides the basis of the development of efficient numerical methods for the wave propagation problems in waveguides. In particular, the problem (3.2)–(3.3) is well adapted to numerical integration by the finite element method and provides excellent robustness properties in terms of the location of the artificial boundaries. As a result we can gain significantly in efficiency by appropriately restricting the computational domain so as to include the areas of interest of the solution.

To be specific, we discretize the problem with the standard/Galerkin finite element method with continuous in Ω piecewise linear functions. The domain Ω is triangulated with triangles of maximal diameter h and nodes on the variable bottom, so that the bottom consists of straight line segments, hence Ω is a polygonal domain. The finite element method is implemented in a Fortran code called FENLCG, which is a modified version of an existing code called FENL^2 concerning an axisymmetric waveguide with two fluid layers; for details we refer to [26]. The latter reference introduced a nonlocal near-field boundary condition incorporating the effect of a point source, which extended an older method, and an associated code called FENL, developed in [20]. For another extension made in order to handle an attenuating sediment layer we refer to [27]. All these codes were extensively validated through comparisons with established coupled normal-mode codes; see, *e.g.*, [1], [26], and [27].

Here, as a proof of concept, we present the outcome of our computations with FENLCG for a downsloping underwater environment, where the bottom profile of the waveguide is given by the function

$$h(x) = \begin{cases} 100 & \text{for } 0 \leq x \leq 300, \\ 150 + 50\cos\frac{\pi(400-x)}{100} & \text{for } 300 < x < 400, \\ 200 & \text{for } x \geq 400. \end{cases}$$

The sound speed is considered to be constant in the water layer, equal to $c_0 = 1500$ m/sec, and the point source of frequency f=25 Hz is located at the vertical y axis at a depth $y_{\mathrm{s}} = 15$ m. For this frequency the wavelength is $\lambda = 60$ m, three modes propagate at the near-field artificial boundary, and six modes propagate at the far-field artificial boundary, *i.e.*, $M_{\mathrm{N}} = 3$ in (2.20)–(2.23) and $M_{\mathrm{F}} = 6$ in (2.9)–(2.10). For the results shown here we have taken into account all the propagating modes in the near- and far-field boundaries and retained the first 12 terms in the series (2.21) and (2.23) and the first two terms in (2.10).

For the verification of the convergence of our code we placed the artificial boundaries at $x_{\mathrm{N}} = 180$ m and $x_{\mathrm{F}} = 460$ m and considered four triangulations of the computational domain, labeled G1 to G4 from coarsest to finest, consisting of 15176, 30215, 60873, 121070 triangles, respectively, corresponding approximately to 26, 37, 53, and 75 meshlengths per wavelength, respectively. For each triangulation we computed the modulus of the solution in an applications logarithmic scale commonly used in underwater acoustics, namely, the transmission loss (TL) defined as $TL(x,y) = -20\log_{10}\left(|u(x,y)|/(0.25|H_0^{(1)}(k)|\right)$, for $(x,y) \in \Omega$, where the Hankel function of the first kind and zero order evaluated at k, $H_0^{(1)}(k)$, acts as a normalization constant measuring the modulus of the field at a distance of 1 m from the source; see [19, Eq. (5.31)].

For a given pair of grids, labeled, say, (a) and (b), and at a fixed receiver depth y_{rd} we computed a 'normalized ℓ_2 TL discrepancy', a measure of the difference between the two solutions $TL^{(a)}(x, y_{rd})$ and $TL^{(b)}(x, y_{rd})$, at $N_{RP} = 281$ equidistant range points $\{(x_i, y_{rd})\}_{i=1}^{N_{RP}}$ in $[x_{\mathrm{N}}, x_{\mathrm{F}}] = [180\text{ m}, 460\text{ m}]$. (The values were computed at these range points by linear interpolation.) The normalized ℓ_2 TL discrepancy (in dB) was defined as the quantity

$$\left(\frac{1}{N_{RP}} \frac{\sum_{i=1}^{N_{RP}} \left|TL^{(a)}(x_i, y_{rd}) - TL^{(b)}(x_i, y_{rd})\right|^2}{\sum_{i=1}^{N_{RP}} \left|TL^{(b)}(x_i, y_{rd})\right|^2}\right)^{1/2}.$$

Table 5.1
Normalized ℓ_2 transmission loss discrepancy.

Depth (m)	G1 vs. G4	G2 vs. G4	G3 vs. G4
15	3.808×10^{-4}	1.290×10^{-4}	5.453×10^{-5}
75	2.751×10^{-4}	1.312×10^{-4}	4.882×10^{-5}

We considered as a reference solution the one obtained with the finest grid G4. In Table 5.1 we report the values of the normalized ℓ_2 TL discrepancies between the results obtained using the grids G1 to G3 and the reference finest grid G4, which are of $O(10^{-4})$ to $O(10^{-5})$ and confirm that the code has converged. In order to give the reader another quantitative measure we note that the maximum discrepancy in absolute value between the TL results obtained with the grid G2 (in which the number of meshlengths per wavelength is approximately half that of G4) and those obtained with the reference grid G4 were found to be 0.124 dB for the receiver depth of 15 m and 0.224 dB for the receiver depth of 75 m.

Next we experimented with the location of the artificial boundaries. Specifically, we considered three domains. The first, labeled Ω_1, is confined in range by the near-field artificial boundary at $x_{\mathrm{N}} = 15$ m and the far-field one at $x_{\mathrm{F}} = 615$ m, in the second (Ω_2) the near- and far-field artificial boundaries were placed at $x_{\mathrm{N}} = 60$ m and $x_{\mathrm{F}} = 520$ m, respectively, while for the third one (Ω_3) at $x_{\mathrm{N}} = 180$ m and $x_{\mathrm{F}} = 460$ m. We have used 66846 elements (triangles) to triangulate Ω_1, 49085 to triangulate Ω_2, and 30215 to triangulate Ω_3. These numbers of elements ensured that in all three domains approximately 37 (average size) meshlengths were contained in a wavelength.

In Fig. 5.1 we present, in the form of two-dimensional TL plots, the results obtained by our code in each of the three domains Ω_1 to Ω_3 (from top to bottom). As one may immediately verify, the location of the artificial boundary does not influence the quality of the approximation. In order to illustrate more detailed information we superimpose in Fig. 5.2 the TL versus range

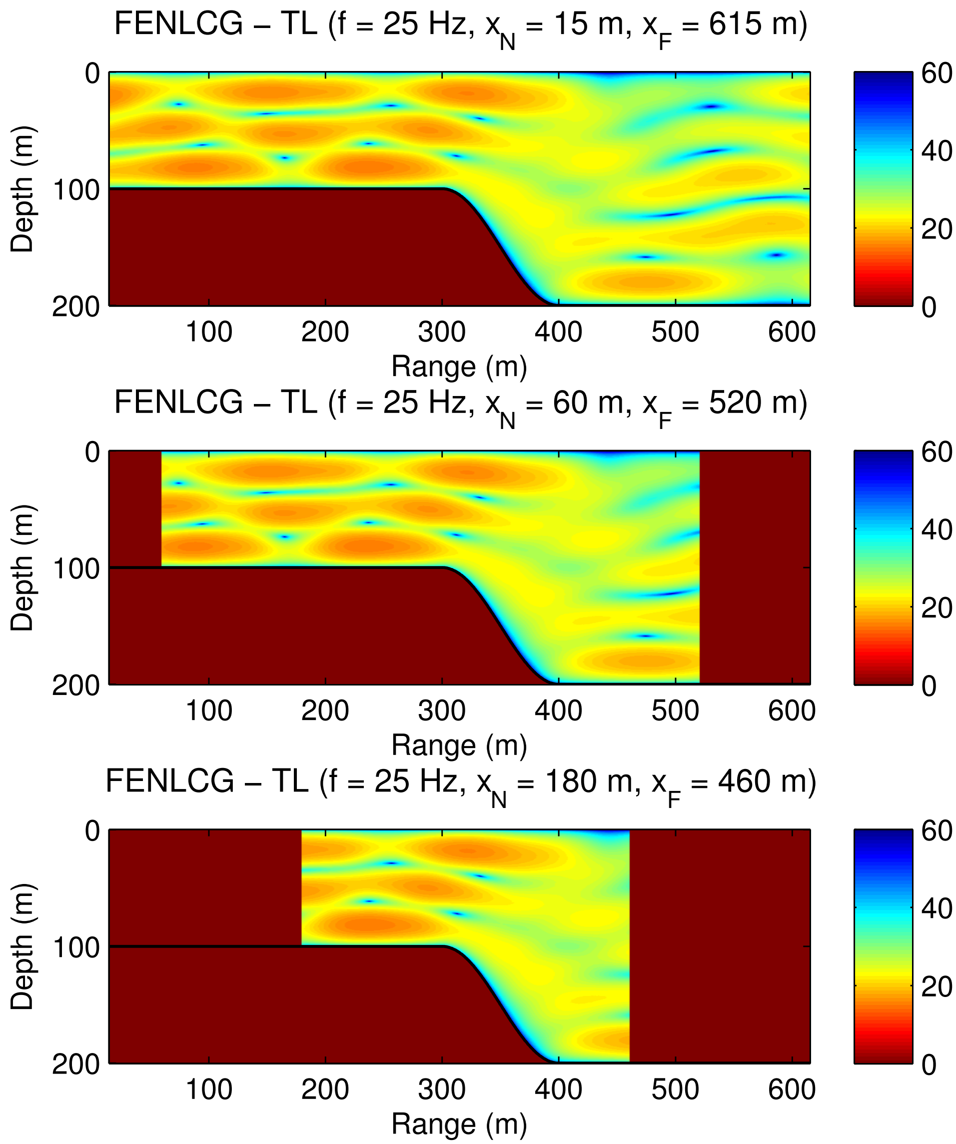

FIG. 5.1. *Two-dimensional TL plots for various positions of the artificial boundaries.*

(x) curves extracted from our results in each of the three domains. In the left subplot we have fixed the (receiver's) depth at the depth of the source, *i.e.*, at 15 m, while in the right subplot the receiver depth is fixed at 75 m. Fig. 5.2 confirms that the results of our code in the three different domains exhibit an excellent agreement in the common region confined between 180 and 460 m in range.

A key conclusion which can be derived by this, and many other simulations that we have performed for various underwater environments, is that these nonlocal boundary conditions posed on the artificial boundaries are proved to be very efficient in truncating the originally infinite domain so as to include just the 'difficult' part of the waveguide, *i.e.*, the part which contains the variable bottom topography in our case. Moreover, placing the artificial boundaries in the vicinity of the irregular bottom topography, such as in Ω_3, allows us to use a much smaller

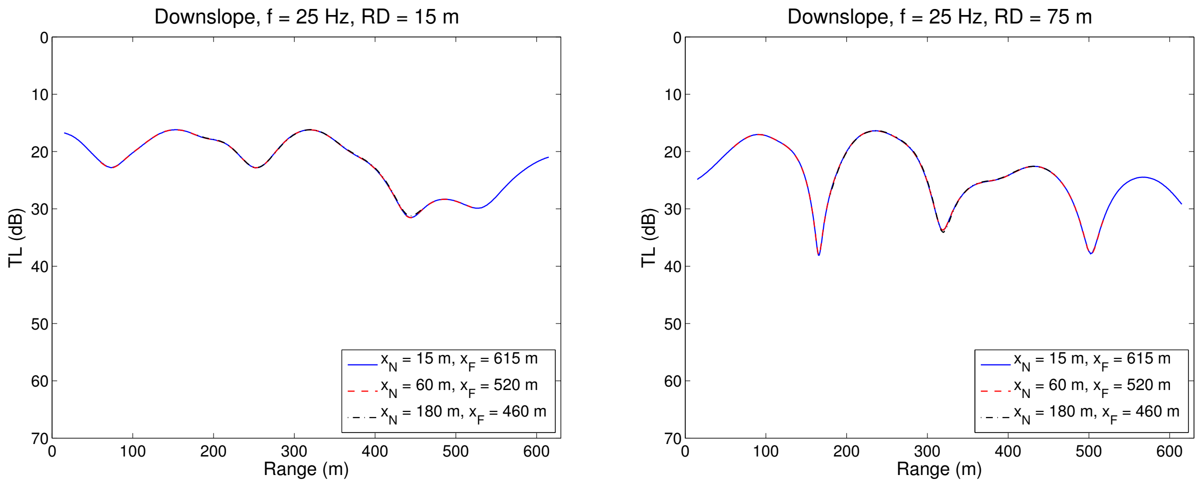

FIG. 5.2. *One-dimensional TL plots for various positions of the artificial boundaries and for receiver depths 15 m (left) and 75 m (right).*

number of elements to triangulate the resulting bounded domain while at the same time keeping a sufficiently large number of meshlengths per wavelength in order to ensure that we capture correctly the acoustic wave propagation behavior. As a final comment, let us note that in the first domain Ω_1 the location of the near-field boundary at $x_{\rm N} = 15$ m ($= \lambda/4$) was chosen in order to satisfy (2.26), thus (2.25) holds, and $(R_1 u, u)_{\Gamma_4} > 0$. On the other hand, in Ω_2 we took $x_{\rm N} = \lambda = 60$ m, in which case (2.24) holds, but $\cot \sqrt{k^2 - \mu_n^{\rm N}}(x_{\rm N} - x_{\rm s}) < 0$, for $n = 1, 2, 3$, hence $(R_1 u, u)_{\Gamma_4} < 0$. Therefore, our numerical results confirm, as discussed in Section 4.2, that (2.25) does not constitute an essential restriction for the location of the near-field artificial boundary.

Acknowledgement. The authors express their thanks to the referees for their useful comments and suggestions.

REFERENCES

[1] G.A. Athanassoulis, K.A. Belibassakis, D.A. Mitsoudis, N.A. Kampanis, V.A. Dougalis, Coupled mode and finite element solutions of underwater sound propagation problems in stratified environments, *J. Comp. Acoust.*, **16** (2008), 83–116.

[2] I. Babuška, Error-bounds for finite element method, *Numer. Math.*, **16** (1970/1971), 322–333.

[3] E. Becache, A.-S. Bonnet-Ben Dhia, G. Legendre, Perfectly matched layers for the convected Helmholtz equation, *SIAM J. Numer. Anal.*, **42** (2004), 409–433.

[4] A. Bendali, Ph. Guillaume, Non-reflecting boundary conditions for waveguides, *Math. Comp.*, **68** (1999), 123–144.

[5] J. Bramble, V. Thomée, Discrete time Galerkin methods for a parabolic boundary value problem, *Ann. Mat. Pura Appl.*, 101 (1974), 115–152.

[6] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.

[7] F. Brezzi, *The inf-sup Condition, the Bubble, and the Subgrid*, http://www.imati.cnr.it/~brezzi/papers/history.pdf (2004).

[8] S.N. Chandler-Wilde, I.G. Graham, S. Langdon, E.A. Spence, Numerical-asymptotic boundary integral methods in high frequency acoustic scattering, *Acta Numer.*, **21** (2012), 89–305.

[9] S.N. Chandler-Wilde, P. Monk, Existence, uniqueness, and variational methods for scattering by unbounded rough surfaces, *SIAM J. Math. Anal.*, **37** (2005), 598–618.

[10] P. Cummings, X. Feng, Sharp regularity coefficient estimates for complex-valued acoustic and elastic Helmholtz equations, *Math. Models Meth. Appl. Sci.*, **16** (2006), 139–160.
[11] V.A. Dougalis, F. Sturm, G.E. Zouraris, On an initial-boundary value problem for a wide-angle parabolic equation in a waveguide with variable bottom, *Math. Meth. Appl. Sci.*, **32** (2009), 1519–1540.
[12] R.B. Evans, A coupled mode solution for the acoustic propagation in a waveguide with stepwise depth variations of a penetrable bottom, *J. Acoust. Soc. Am.*, **74** (1983), 188–195.
[13] G.J. Fix, S.P. Marin, Variational methods for underwater acoustic problems, *J. Comput. Phys.*, **28** (1978), 253–270.
[14] D. Givoli, *Numerical methods for problems in infinite domains*, Elsevier, New York, 1992.
[15] C.I. Goldstein, A finite element method for solving Helmholtz type equations in waveguides and other unbounded domains, *Math. Comp.*, **39** (1982), 309–324.
[16] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston, 1985.
[17] T. Hagstrom, A. Mar-Or, D. Givoli, High-order local absorbing conditions for the wave equation: Extensions and improvements, *J. Comput. Phys.*, **227** (2008), 3322–3357.
[18] F. Ihlenburg, *Finite Element Analysis of Acoustic Scattering*, Appl. Math. Sci. 132, Springer, 1998.
[19] F.B. Jensen, W.A. Kuperman, M.B. Porter, H. Schmidt, *Computational Ocean Acoustics*, Springer, New York, 2000.
[20] N.A. Kampanis, V.A. Dougalis, A finite element code for the numerical solution of the Helmholtz equation in axially symmetric waveguides with interfaces, *J. Comp. Acoust.*, **7** (1999), 83–110.
[21] C.M. Linton, The Green's function for the two-dimensional Helmholtz equation in periodic domains, *J. Engng. Math.*, **33** (1998), 377–402.
[22] J.-L. Lions, E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Vol. 1, Springer, New York, 1972.
[23] Ch. Makridakis, F. Ihlenburg, I. Babuška, Analysis and finite element methods for a fluid-solid interaction problem in one dimension, *Math. Models Meth. Appl. Sci.*, **6** (1996), 1119–1141.
[24] J. Melenk, *On Generalized Finite Element Methods*, Ph.D. thesis, University of Maryland, College Park, MD, 1995.
[25] J.M. Melenk, S. Sauter, Convergence analysis for finite element discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions, *Math. Comp.*, **79** (2010), 1871–1914.
[26] D.A. Mitsoudis, Near- and far-field boundary conditions for a finite element method for the Helmholtz equation in axisymmetric problems of underwater acoustics, *Acta Acust. united with Ac.*, **93** (2007), 888–898.
[27] D.A. Mitsoudis, M. Plexousakis, A finite element method with nonlocal boundary conditions for the Helmholtz Equation with complex wavenumber in stratified waveguides, *Acta Acust. united with Ac.*, **95** (2009), 753–756.
[28] C.S. Morawetz, D. Ludwig, An inequality for the reduced wave operator and the justification of geometrical optics, *Comm. Pure Appl. Math.*, **21** (1968), 187–203.
[29] L. Nirenberg, Remarks on strongly elliptic partial differential equations, *Comm. Pure Appl. Math.*, **8** (1955), 649–675.
[30] N.A. Phillips, A coordinate system having some special advantages for numerical forecasting, *J. Meteor.*, **14** (1957), 184–185.
[31] I. Singer, E. Turkel, A perfectly matched layer for the Helmholtz equation in a semi-infinite strip, *J. Comput. Phys.*, **201** (2004), 439–465.
[32] S.V. Tsynkov, Numerical solution of problems on unbounded domains. A review, *Appl. Numer. Math.*, **27** (1998), 465–532.
[33] K. Yosida, *Functional Analysis*, Springer, New York, 1980.